University at Buffalo **Spring Semester 2014**
Yorck Sommerhäuser **MTH 819: Sheet 2**

Hopf Algebras

Problem 1: Suppose that $q \in K$ is an element of the base field that is different from 0 and 1. For a natural number $n \in \mathbb{N}$, we define the q-number

$$(n)_q := \frac{q^n - 1}{q - 1} = \sum_{k=0}^{n-1} q^k$$

Furthermore, we define the q-factorial as $(n)_q! := (1)_q(2)_q(3)_q \dots (n)_q$, with the convention that $(0)_q! := 1$. It may very well happen that q-numbers and q-factorials are zero; this happens if and only if q is a root of unity. Even in this case, we define the q-binomial coefficient, or Gaussian binomial coefficient, as

$$\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}$$

as long as the denominator of this expression is nonzero. However, we have in any case by definition that

$$\binom{n}{0}_q = \binom{n}{n}_q := 1$$

even if $(n)_q! = 0$. Furthermore, we define $\binom{0}{0}_q := 1$.

1. Show the q-Pascal identity
$$\binom{n}{k}_q = q^{n-k}\binom{n-1}{k-1}_q + \binom{n-1}{k}_q$$

2. Suppose that A is an algebra and that $a, b \in A$ q-commute in the sense that $ba = qab$. Show the q-binomial theorem
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}$$

3. In this situation, suppose that q is a primitive n-th root of unity. Show that
$$(a+b)^n = a^n + b^n$$

(6 points)

Problem 2: Suppose that $q \in K$ is a primitive n-th root of unity. Let T be the algebra generated by g and x subject to the relations

$$g^n = 1 \qquad x^n = 0 \qquad xg = qgx$$

1. Show that there exist unique algebra homomorphisms $\Delta : T \to T \otimes T$ and $\varepsilon : T \to K$ such that

   $$\Delta(g) = g \otimes g \qquad \Delta(x) = 1 \otimes x + x \otimes g \qquad \varepsilon(g) = 1 \qquad \varepsilon(x) = 0$$

   and a unique algebra antihomomorphism $S : T \to T$ satisfying

   $$S(g) = g^{n-1} \qquad S(x) = -xg^{n-1}$$

   (Note that $g^{n-1} = g^{-1}$, the inverse of g.) (2 points)

2. Show that Δ, ε, and S make T into a Hopf algebra, the so-called Taft algebra. (3 points)

Problem 3: Let C be a cyclic group of order n with generator c, and let $K[t]/(t^n)$ be the quotient of the polynomial algebra $K[t]$ by the principal ideal (t^n). We denote the residue class of t by τ. The elements $c^i \otimes \tau^j$ for indices in the range $i, j = 0, \ldots, n-1$ form a basis of $K[C] \otimes K[t]/(t^n)$. We define a product on this space by defining it on basis elements via

$$(c^i \otimes \tau^j)(c^k \otimes \tau^l) = q^{jk} c^{i+k} \otimes \tau^{j+l}$$

and extending bilinearly.

1. Show that this product is associative and that $1 \otimes 1 = c^0 \otimes t^0$ is a unit element. (2 points)
2. Show that $K[C] \otimes K[t]/(t^n)$ is isomorphic to T. (2 points)
3. Conclude that the elements $g^i x^j$ for $i, j = 0, \ldots, n-1$ form a basis of T. (1 point)

Problem 4: Suppose that H is a Hopf algebra with antipode S. Suppose furthermore that H^{op} is also a Hopf algebra with antipode S'. Show that $S \circ S' = \mathrm{id}_H = S' \circ S$. (4 points)

Due date: Tuesday, February 11, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.