University at Buffalo
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Spring Semester 2014
MTH 819: Sheet 1

Hopf Algebras

Problem 1: Suppose that V and W are vector spaces over the field K. Show that every tensor $x \in V \otimes W$ can be written as a sum

$$x = \sum_{i=1}^{k} v_i \otimes w_i$$

of decomposable tensors $v_i \otimes w_i$ with the property that both $v_1, \dots, v_k$ and $w_1, \dots, w_k$ are linearly independent. (4 points)

Problem 2: Suppose that V and W are vector spaces over the field K.

1. If $(v_i)_{i \in I}$ is a basis of V, where I is a not necessarily finite index set, show that every tensor $x \in V \otimes W$ can be written uniquely in the form

   $$x = \sum_{i \in I} v_i \otimes w_i$$

   where only finitely many w_i are nonzero. (4 points)

2. If in addition $(w_j)_{j \in J}$ is a basis of W, where J is a not necessarily finite index set, show that every tensor $x \in V \otimes W$ can be written uniquely in the form

   $$x = \sum_{i \in I} \sum_{j \in J} \lambda_{ij} v_i \otimes w_j$$

   where only finitely many coefficients $\lambda_{ij} \in K$ are nonzero. (2 points)

Problem 3: Suppose that V and W are vector spaces over the field K, and that $V^* = \mathrm{Hom}_K(V, K)$ and $W^* = \mathrm{Hom}_K(W, K)$ are the corresponding dual spaces. If $f \in V^*$ and $g \in W^*$, the mapping

$$V \times W \to K,\ (v, w) \mapsto f(v)g(w)$$

is bilinear, and therefore induces a linear map $V \otimes W \to K$, i.e., an element of the dual space $(V \otimes W)^*$, which is, in an ambiguous fashion, also denoted by $f \otimes g$.

1. Show that the map

$$V^* \times W^* \to (V \otimes W)^*, \ (f, g) \mapsto f \otimes g$$

   is bilinear and therefore induces a canonical map

$$c : V^* \otimes W^* \to (V \otimes W)^*$$

   satisfying $c(f \otimes g) = f \otimes g$. (Recall that the notation $f \otimes g$ is ambiguous.) (2 points)

2. Show that c is injective. (3 points)

Problem 4:

1. Show that c is surjective if V or W are finite-dimensional. (2 points)

2. Show that c is not surjective if both V and W are infinite-dimensional. (3 points)

Due date: Tuesday, February 4, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.