University at Buffalo
Yorck Sommerhäuser

Spring Semester 2014
MTH 819: Sheet 11

Hopf Algebras

Problem 1: Suppose that K is a field whose characteristic is different from 2 and that $\iota \in K$ is a primitive fourth root of unity. Consider the Hopf algebra H defined in Problem 1 on Sheet 10.

1. Show that there are four algebra homomorphisms $\omega_0, \omega_1, \omega_2, \omega_3$ from H to K, of which the first is the counit $\omega_0 = \varepsilon$ defined in Problem 1 on Sheet 10 and the remaining three are given on generators by

$$\begin{array}{lll} \omega_1(x) = 1 & \omega_1(y) = 1 & \omega_1(z) = -1 \\ \omega_2(x) = -1 & \omega_2(y) = -1 & \omega_2(z) = \iota \\ \omega_3(x) = -1 & \omega_3(y) = -1 & \omega_3(z) = -\iota \end{array}$$

2. Show that there are no other algebra homomorphisms from H to K.

3. Show that the group of group-like elements $G(H^*)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (6 points)

Problem 2: We continue to use the assumptions and notations from Problem 1.

1. Show that there is a representation $\rho : H \to M(2\times 2, K)$ with the property that

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. Show that ρ is irreducible.

3. Show that every two-dimensional irreducible representation is isomorphic to ρ. (6 points)

Problem 3: For the Hopf algebra H considered in Problem 1, show that the elements $x^i y^j z^k$ for $i, j, k \in \{0, 1\}$ form a basis of H. Use this to compute the dimension of H. (4 points)

Problem 4: For a finite-dimensional Hopf algebra H, the Drinfel'd double $D(H)$ is defined as the algebra with underlying vector space $D(H) := H^* \otimes H$, multiplication

$$(\varphi \otimes h)(\varphi' \otimes h') = \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(h_{(1)})\ \varphi\varphi'_{(2)} \otimes h_{(2)}h'$$

and unit element $\varepsilon \otimes 1$. Show that these definitions make $D(H)$ into an associative unital algebra. (4 points)

Due date: Tuesday, April 22, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.