University at Buffalo
Yorck Sommerhäuser

Spring Semester 2014
MTH 819: Sheet 10

Hopf Algebras

Problem 1: Suppose that K is a field whose characteristic is different from 2. Let H be the algebra with generators x, y, z subject to the relations

$$x^2 = y^2 = 1 \qquad z^2 = \frac{1}{2}(1 + x + y - xy) \qquad xy = yx \qquad xz = zy \qquad yz = zx$$

1. Show that there are unique algebra homomorphisms

   $$\Delta : H \to H \otimes H \qquad \varepsilon : H \to K \qquad S : H \to H^{\mathrm{op}}$$

   with the property that

   $$\Delta(x) = x \otimes x \qquad \Delta(y) = y \otimes y \qquad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$

   as well as $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$ and $S(x) = x$, $S(y) = y$, $S(z) = z$.

2. Show that these structures make H into a Hopf algebra. (6 points)

Problem 2: For the Hopf algebra in Problem 1, define

$$\Lambda := (1 + x)(1 + y)(1 + z)$$

1. Show that Λ is a two-sided integral.
2. Deduce that H is semisimple. (4 points)

Problem 3: Suppose that G is a finite group and that $H = K[G]^*$ is the dual group ring considered in Problem 1 on Sheet 3. As there, we denote the dual basis elements of the group elements by p_g. Let

$$R = \sum_{g,h \in G} \theta(g, h) p_g \otimes p_h \in H \otimes H$$

be any tensor. Show that H is quasitriangular with universal R-matrix R if and only if G is abelian and θ is a bicharacter, i.e., a map from $G \times G$ to the multiplicative group $K^\times$ that satisfies

$$\theta(gg', h) = \theta(g, h)\theta(g', h) \qquad \text{and} \qquad \theta(g, hh') = \theta(g, h)\theta(g, h')$$

for all $g, g', h, h' \in G$. (4 points)

Problem 4: Suppose that H is a quasitriangular Hopf algebra with universal R-matrix R and Drinfel'd element u. Show that

$$\Delta(u) = (u \otimes u)(R_{21}R)^{-1} = (R_{21}R)^{-1}(u \otimes u)$$

(6 points)

Due date: Tuesday, April 15, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.