University at Buffalo **Spring Semester 2015**
Yorck Sommerhäuser **MTH 620: Sheet 8**

Algebra II

Problem 1: Suppose that E is 'the' splitting field of $f(x) = x^3 - 2$ over $F = \mathbb{Q}$.

1. Find $n := [E : F]$. (2 points)
2. Make an $n \times 3$-table in which you describe the action of each element of $G := \mathrm{Gal}(E/F)$ on the three roots of f explicitly. (1 point)
3. Make a list of all (isomorphism types of) groups of order n, and decide which one of these is isomorphic to G. (1 point)
4. Use the fundamental theorem of Galois theory (cf. E. Artin, Galois theory, Theorem 16, page 46) to find all intermediate fields of our Galois extension $F \subset E$. (2 points)

Problem 2: For a field F, suppose that $f(x)$ and $g(x)$ are relatively prime polynomials in $F[x]$. Show that the polynomial $yg(x) - f(x)$ is irreducible in $F[x, y]$ and in $F(x)[y]$. (2 points)

Problem 3: Suppose that F is a field. A rational function $u = u(x) \in F(x)$ can be written in the form

$$u(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are relatively prime polynomials in $F[x]$. In the sequel, we assume that $u(x) \notin F$, i.e., that $u(x)$ is not constant.

1. Show that u is transcendental over F. (An element is called transcendental if it is not algebraic.) (2 points)
2. Show that $ug(y) - f(y) \in F(u)[y]$ is the minimum polynomial of x over $F(u)$. (2 points)
3. Show that $[F(x) : F(u)] = \max(\deg(f), \deg(g))$. (2 points)

Problem 4: Suppose that F is a field, that $G = \mathrm{Gal}(F(x)/F)$ is the Galois group, and that $E \subset F(x)$ is the fixed field of G.

1. Show that every automorphism $\sigma \in G$ is a fractional linear substitution, i.e., that there are $a, b, c, d \in F$ with $ad - bc \neq 0$ so that
$$\sigma(u(x)) = u\left(\frac{ax + b}{cx + d}\right)$$
(2 points)
2. Show that $E = F$ if F is infinite. (2 points)
3. Show that $E \neq F$ if F is finite. (2 points)

Due date: Wednesday, April 15, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.