University at Buffalo
Yorck Sommerhäuser

Spring Semester 2015
MTH 620: Sheet 7

Algebra II

Problem 1: Suppose that a and b are positive rational numbers that are not squares of rational numbers. Suppose that also ab is not the square of a rational number. For the fields $F := \mathbb{Q}$ and $E := \mathbb{Q}(\sqrt{a}, \sqrt{b}) \subset \mathbb{R}$, show that $[E : F] = 4$. (4 points)

Problem 2: For the field $F := \mathbb{Q}$ of rational numbers, consider the extension field $E := \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \mathbb{R}$.

1. Show that $F \subset E$ is a Galois extension. (2 points)
2. In view of the preceding problem, the Galois group $\mathrm{Gal}(E/F)$ has order 4. According to Problem 3 on Sheet 2 from last semester, there are two isomorphism types of such groups. Find the isomorphism type of $\mathrm{Gal}(E/F)$, and find the images of both $\sqrt{2}$ and $\sqrt{3}$ under all four automorphisms in $\mathrm{Gal}(E/F)$. (2 points)
3. Use the fundamental theorem of Galois theory (cf. E. Artin, Galois theory, Theorem 16, page 46) to find all intermediate fields of our Galois extension $F \subset E$. (2 points)

Problem 3: As in the last problem, let $F := \mathbb{Q}$ and $E := \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \mathbb{R}$.

1. Show that $E := \mathbb{Q}(\sqrt{2} + \sqrt{3})$. (2 points)
2. Find the minimum polynomial of $\sqrt{2} + \sqrt{3}$ over F. (2 points)

(Hint: For the second part, proceed as in the proof of the lemma in E. Artin, Galois theory, page 46.)

Problem 4: For a field K, we consider the field $E := K(x_1, \ldots, x_n)$ of rational functions in n indeterminates. Besides the elementary symmetric functions

$$b_k(x_1, \ldots, x_n) = (-1)^k a_k(x_1, \ldots, x_n)$$

(cf. E. Artin, Galois theory, page 39), we introduce the power sums

$$s_k(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^k = x_1^k + \cdots + x_n^k$$

Show Newton's recursion formula

$$s_j + \sum_{i=1}^{j-1} a_i s_{j-i} + j a_j = s_j + \sum_{i=1}^{j-1} (-1)^i b_i s_{j-i} + (-1)^j j b_j = 0$$

für $j = 1, \ldots, n$.

(Hint: Apply the formal differential operator

$$\Delta_j = t^{j+1} \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i^{j+1} \frac{\partial}{\partial x_i}$$

to the polynomial

$$\prod_{i=1}^{n} (t - x_i) = \sum_{j=0}^{n} a_{n-j}(x_1, \ldots, x_n) t^j$$

in the polynomial ring $K[x_1, \ldots, x_n, t]$ and differentiate the left-hand side according to the product rule and the right-hand side explicitly. Then compare the coefficient of t^n on both sides. (6 points)

Due date: Wednesday, April 8, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.