University at Buffalo
Yorck Sommerhäuser

Spring Semester 2015
MTH 620: Sheet 6

Algebra II

Problem 1: For the field $F := \mathbb{Q}$ of rational numbers, the field $E := \mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R}$ is an extension field.

1. Find $[E : F]$. (2 points)
2. Find the Galois group $\mathrm{Gal}(E/F)$. (3 points)
3. Find a field extension $F \subset E$ with the property that the fixed field of the Galois group $\mathrm{Gal}(E/F)$ is strictly larger than F (cf. E. Artin, Galois theory, page 38). (1 point)

Problem 2: For field F, the field $E := F(x)$ of rational functions, i.e., the quotient field of the polynomial ring $F[x]$, is an extension field of F.

1. For an invertible 2×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$$

show that there is an automorphism $\sigma(A) \in \mathrm{Gal}(E/F)$ with the property

$$(\sigma(A)(f))(x) = f\left(\frac{ax+b}{cx+d}\right)$$

for all $f = f(x) \in E$. (Such an automorphism $\sigma(A)$ is called a fractional linear substitution.) (1 point)

2. Show that

$$\sigma : \mathrm{GL}(2, F) \to \mathrm{Gal}(E/F),\ A \mapsto \sigma(A)$$

is a group homomorphism. (2 points)

3. Show that the group N of nonzero scalar multiples of the identity matrix, which is a normal subgroup of $\mathrm{GL}(2, F)$, is contained in the kernel of σ. Therefore, σ factors over the corresponding quotient group

$$\mathrm{PGL}(2, F) := \mathrm{GL}(2, F)/N$$

the so-called projective linear group (in dimension 2 over F). (1 point)

Problem 3:

1. Show that there is a group homomorphism $\rho : S_3 \to \mathrm{PGL}(2, F)$ from the symmetric group on three letters to the projective linear group in dimension 2 over F with the property

$$\rho((1,2)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad \rho((2,3)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(2 points)

2. List the images of all six elements of S_3 under ρ explicitly. (1 point)

3. Using the homomorphism σ from Problem 2, list the rational function $((\sigma \circ \rho)(g))(x)$ for all $g \in S_3$ explicitly. (1 point)

(Remarks: This problem should be compared with E. Artin, Galois theory, § II.G, page 38. Note that the statement of the problem is not totally accurate, as we should have used their residue classes modulo N rather than the matrices themselves in the first part, and the factorization of σ modulo N rather than σ itself in the third part. In the first part, it is very helpful to use a suitable presentation of S_3. For this, Problem 2 on Sheet 4 from last semester may be helpful.)

Problem 4: If $F = \mathbb{Q}$ is the field of rational numbers, let E be 'the' splitting field of the polynomial $p(x) := x^3 - 1 \in F[x]$.

1. Find the degree $[E : F]$ of the extension. (3 points)

2. Find the order $|\,\mathrm{Gal}(E/F)|$ of the Galois group. (3 points)

(Remark: By E. Artin, Galois theory, Theorem 10, the field E is essentially unique.)

Due date: Wednesday, April 1, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.