University at Buffalo **Spring Semester 2015**
Yorck Sommerhäuser **MTH 620: Sheet 4**

Algebra II

Problem 1: Let R be a commutative unital ring. Recall that greatest common divisors are defined in Hungerford, Definition 3.10, page 140. Similarly, an element $l \in R$ is called a common multiple of the subset $X \subset R$ if $x|l$ for all $x \in X$. A common multiple l of X is called a least common multiple if every common multiple l' of X is divisible by l.
Now let $a, b \in R$ and $X = \{a, b\}$. Show that, if $d, l \in R$ are elements that satisfy the relations

$$(a) + (b) = (d) \qquad\qquad (a) \cap (b) = (l)$$

for principal ideals, then d is a greatest common divisor and and l is a least common multiple for X. (2 points)

Problem 2: Let R be a commutative unital ring and $S \subset R$ a multiplicatively closed set that contains 1, but does not contain 0. Then the canonical homomorphism φ takes the form

$$\varphi : R \to S^{-1}R, \ r \mapsto \frac{r}{1}$$

For an ideal $J \subset S^{-1}R$, we define the ideal $J \cap R := \varphi^{-1}(J)$, even if φ is not injective.

1. For an ideal $I \subset R$, show that

$$S^{-1}I := \left\{ \frac{r}{s} \mid r \in I \right\}$$

is an ideal in $S^{-1}R$, which is prime if I is prime and does not intersect S. (2 points)

2. For an ideal $J \subset S^{-1}R$, show that $J = S^{-1}(J \cap R)$. (2 points)

3. Show that the correspondences $I \mapsto S^{-1}I$ and $J \mapsto J \cap R$ are bijective correspondences between the prime ideals of $S^{-1}R$ and the prime ideals of R that do not intersect S. (2 points)

Problem 3: Let R be a commutative unital ring, $P \subset R$ a prime ideal, and $S := R \setminus P$.

1. Show that R/P is an integral domain. (1 point)
2. Show that $S^{-1}P$ is a maximal ideal of $R_P := S^{-1}R$. (1 point)
3. Show that $R_P/(S^{-1}P)$ is isomorphic to the quotient field of R/P. (4 points)

Problem 4: Suppose that R is an integral domain with quotient field K.

1. If $P \subset R$ is a prime ideal, use the universal property of localizations to show that there is a canonical map $\iota_P : R_P \to K$. (1 point)
2. Show that ι_P is injective. In this way, we can consider R_P as those fractions in the quotient field whose denominator is not in P (for some, but not all, representations of this element as a fraction). (1 point)
3. Show that
$$\bigcap_{P \text{ prime ideal}} R_P = R$$
where R_P is considered as a subset of K in the sense of the second part.

   (Hint: It already suffices to take the intersection over all maximal ideals instead of all prime ideals.) (4 points)

Due date: Wednesday, March 4, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.