University at Buffalo **Spring Semester 2015**
Yorck Sommerhäuser **MTH 620: Sheet 3**

Algebra II

Problem 1: The set of Gaussian integers is defined as

$$R := \{a + ib \mid a, b \in \mathbb{Z}\}$$

For a Gaussian integer $z = a + ib$, we define its norm as

$$N(z) := z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$$

1. Show that R is a unital subring of the complex numbers. (0.5 points)
2. Show that the norm is multiplicative in the sense that $N(zw) = N(z)N(w)$ for all $z, w \in R$. (0.5 points)
3. Show that $z \in R$ is a unit if and only if $N(z) = 1$. (1 point)
4. Determine the set of all units in the Gaussian integers R. (1 point)
5. Show that the ring of Gaussian integers is Euclidean with respect to the norm function. (3 points)

Problem 2: Consider the set

$$R := \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$$

For an element $z = a + b\sqrt{-5} \in R$, we define its norm as

$$N(z) := z\bar{z} = (a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2$$

1. Show that R is a unital subring of the complex numbers. (0.5 points)
2. Show that the norm is multiplicative in the sense that $N(zw) = N(z)N(w)$ for all $z, w \in R$. (0.5 points)
3. Show that $z \in R$ is a unit if and only if $N(z) = 1$. (0.5 points)
4. Determine the set of all units in R. (0.5 points)
5. Show that $9 = 3 \cdot 3$ and

   $$9 = (2 + \sqrt{-5})(2 - \sqrt{-5})$$

   are two factorizations of 9 into irreducible elements. (2 points)

6. Show that neither $2+\sqrt{-5}$ nor $2-\sqrt{-5}$ is associated to 3. (1.5 points)
7. Decide whether R is a unique factorization domain. (0.5 points)

Problem 3: Consider the set

$$R := \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

For an element $z = a + b\sqrt{2} \in R$, we define its norm as

$$N(z) := (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$$

1. Show that R is a unital subring of the real numbers. (0.5 points)
2. Show that the norm is multiplicative in the sense that $N(zw) = N(z)N(w)$ for all $z, w \in R$. (0.5 points)
3. Show that $z \in R$ is a unit if and only if $N(z) = \pm 1$. (1 point)
4. Show that $\varepsilon := 1 + \sqrt{2}$ is a unit in R. (1 point)
5. Decide whether R contains finitely or infinitely many units. (1 point)

(Remark: It can be shown that every unit in R is a power of ε or the negative of a power of ε. This question is related to the so-called Pell equation and the Dirichlet unit theorem.)

Problem 4: Suppose that R is a Euclidean ring with respect to φ, and that $a, b \in R$ are nonzero. We define recursively a (possibly finite or infinite) sequence of elements by setting $r_0 := a$, $r_1 := b$, and the recursion relation

$$r_{i-1} = q_i r_i + r_{i+1}$$

where $r_{i+1} = 0$ or $\varphi(r_{i+1}) < \varphi(r_i)$. Here, r_{i-1} and r_i are given, and q_i and r_{i+1} exist according to the definition of a Euclidean ring. The recursion stops if one of the remainders, say r_{n+1}, is zero.

1. Show that the sequence is in fact finite; i.e., the recursion indeed stops. (1 point)
2. Show that the last nonzero remainder, say r_n, is a greatest common divisor of a and b. (3 points)

Due date: Wednesday, February 25, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.