**University at Buffalo**
**Yorck Sommerhäuser**

**Spring Semester 2015**
**MTH 620: Sheet 2**

# Algebra II

**Problem 1:** Suppose that $R$ is a commutative ring, and define its nilradical to be the set

$$N := \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{N}\}$$

Show that $N$ is an ideal of $R$. (4 points)

**Problem 2:** Consider the subset

$$\mathbb{H} := \{\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mid z, w \in \mathbb{C}\}$$

of the set $M(2 \times 2, \mathbb{C})$ of complex $2 \times 2$-matrices, and its elements

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad K := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

1. Show that $\mathbb{H}$ is a four-dimensional real subspace of $M(2 \times 2, \mathbb{C})$ with basis $E$, $I$, $J$, $K$. (Note that $\mathbb{H}$ is not a complex subspace.) (1 point)
2. Show that $\mathbb{H}$ is a subring of $M(2 \times 2, \mathbb{C})$ with the same unit element. (1 point)
3. Show that every nonzero element of $\mathbb{H}$ is invertible. (3 points)
4. Make a table that contains all the sixteen possible products of the elements $E$, $I$, $J$, and $K$. (1 point)

**Problem 3:** Let $R$ be a unital ring and $I_1, \ldots, I_n$ be (two-sided) ideals with the property that $I_i + I_j = R$ whenever $i \neq j$. Show that the map

$$R \to (R/I_1) \times (R/I_2) \times \ldots \times (R/I_n), \; x \mapsto (\bar{x}, \bar{x}, \ldots, \bar{x})$$

is a surjective ring homomorphism, where the right-hand side is a ring with respect to componentwise addition and multiplication. (6 points)

**Problem 4:** Suppose that $R$ is a not necessarily unital ring. On the abelian group $S := R \times \mathbb{Z}$ with componentwise addition, define the multiplication

$$(r_1, n_1)(r_2, n_2) := (r_1 r_2 + n_2 r_1 + n_1 r_2, n_1 n_2)$$

Show that $S$ is a unital ring with unit element $(0, 1)$ and that

$$f : R \to S, \ r \mapsto (r, 0)$$

is a ring homomorphism. (4 points)

Due date: Wednesday, February 18, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.