University at Buffalo
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Spring Semester 2015
MTH 620: Sheet 1

Algebra II

Problem 1: Show that every element in a free group can be uniquely written as a reduced word in the generators.

(Hint: Recall that we did this in class: If X was the set of generators, we defined $Y := X \times \{1, -1\}$ and said that $(y_1, \dots, y_n) \in Y^n$ is a reduced word of length n if we have, for all $i = 1, \dots, n-1$, that

$$\begin{aligned} &\text{if } y_i = (x, 1), \text{ then } y_{i+1} \neq (x, -1) \\ &\text{if } y_i = (x, -1), \text{ then } y_{i+1} \neq (x, 1) \end{aligned}$$

We denoted the set of reduced words of length n by Y_n and the set of all reduced words by $W := \bigcup_{n=0}^{\infty} Y_n$. Finally, we defined

$$\rho : W \to \tilde{F}(X),\ (y_1, \dots, y_n) \mapsto \overline{\iota(y_1)} \cdot \ldots \cdot \overline{\iota(y_n)}$$

The assertion then means that ρ is bijective.) (4 points)

Problem 2: Suppose that $X = \{a, b\}$ is a set with two elements. We define

$$\varphi : X \to \mathrm{GL}(2, \mathbb{R}),\ a \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

By the universal property of free groups, there is a group homomorphism

$$\psi : \tilde{F}(X) \to \mathrm{GL}(2, \mathbb{R})$$

with the property that $\psi(\tilde{\iota}(a)) = \varphi(a)$ and $\psi(\tilde{\iota}(b)) = \varphi(b)$. Show that ψ is injective.

(Hint: Consider the sets

$$U := \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| > |y|\} \qquad V := \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid |x| < |y|\}$$

For $n \neq 0$, show that $\varphi(a)^n V \subset U$ and $\varphi(b)^n U \subset V$. Conclude that, if $w \in \tilde{F}(X)$ is a reduced word that begins and ends with a nontrivial power of a, then $\psi(w)(V) \subset U$, so that $\psi(w) \neq 1$. Reduce other forms of reduced words to this case.) (6 points)

Problem 3: Decide whether the group

$$\langle x, y \mid x^{-1}y^2x = y^3, y^{-1}x^2y = x^3\rangle$$

is finite or infinite. If it is finite, find its order. (6 points)

Problem 4: Decide whether the group

$$\langle x, y \mid x^{-1}yx = y^{-1}, y^{-1}xy = x^{-1}, x^2 = y^2\rangle$$

is finite or infinite. If it is finite, find its order. (4 points)

Due date: Friday, February 13, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.