University at Buffalo
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Spring Semester 2015
MTH 620: Sheet 10

Algebra II

Problem 1: Suppose that E is 'the' splitting field of $f(x) = x^8 - 2$ over $F = \mathbb{Q}$.

1. Find $n := [E : F]$. (2 points)

2. Find two elements of the Galois group $\mathrm{Gal}(E/F)$ that generate the group. Make a table in which you express all n elements of the Galois group explicitly in terms of these generators. (2 points)

3. The Galois group $\mathrm{Gal}(E/F)$ is a 2-group with a cyclic maximal subgroup. Such groups were classified in Kurzweil/Stellmacher, 5.3.2, page 108. Determine the type of the Galois group $\mathrm{Gal}(E/F)$ in terms of this classification. (2 points)

(Remark: As always, prove all your claims in complete detail.)

Problem 2: Suppose that $f(x) \in F[x]$ is a separable polynomial and that $x_1, \ldots, x_n$ are its roots in its splitting field E. Define

$$\delta := \prod_{\substack{i,j=1 \\ i<j}}^{n} (x_i - x_j)$$

We view the Galois group G of E over F as a subgroup of the group S_n of permutations of the roots (cf. E. Artin, Galois theory, page 76). In particular, we can say that an element of the Galois group is even or odd.

1. Show for all $\sigma \in G$ that $\sigma(\delta) = \epsilon(\sigma)\delta$, where $\epsilon(\sigma)$ is the sign of the permutation corresponding to $\sigma \in G$.

2. $\Delta := \delta^2$, the so-called discriminant of the polynomial, is contained in F.

3. The Galois group is contained in the alternating group if and only if $\delta \in F$.

(4 points)

Problem 3: Suppose that $f(x) \in F[x]$ is a separable irreducible polynomial of degree 3 with splitting field E. Using the quantity Δ defined in the preceding problem, show that the following conditions are equivalent:

1. $[E : F] = 3$
2. $\mathrm{Gal}(E/F) \cong A_3$
3. Δ is the square of an element in F.

In case that these equivalent conditions are not met, show that $[E : F] = 6$ and $\mathrm{Gal}(E/F) \cong S_3$. (4 points)

Problem 4: Suppose that F is a field whose characteristic is not 2 or 3.

1. If $g(x) = x^3 + ux^2 + vx + w \in F[x]$ is a cubic polynomial, show that the polynomial $f(x) = g(x - \frac{u}{3})$ has the form $f(x) = x^3 + px + q$, i.e., has no quadratic term. (This transformation is a special case of a so-called Tschirnhaus transformation.) (1 point)

2. Prove Cardano's formula for cubic equations: The roots of the polynomial
$$x^3 + px + q = 0$$
with coefficients $p, q \in F$ can be written in the form
$$a + b \qquad \xi a + \xi^2 b \qquad \xi^2 a + \xi b$$
in a field $E = F(\xi, \delta, a, b)$, where ξ is a primitive third root of unity, $\delta := \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, and the third roots
$$a := \sqrt[3]{-\frac{q}{2} + \delta} \qquad b := \sqrt[3]{-\frac{q}{2} - \delta}$$
which can be chosen in three ways each, are chosen so that $ab = -\frac{p}{3}$. (Hint: Make the ansatz $x = a + b$. Inserting this ansatz into the the equation, one obtains the sufficient conditions
$$3ab = -p \qquad a^3 + b^3 = -q$$
Use these equations to set up the quadratic equation
$$(y - a^3)(y - b^3) = 0$$
for a^3 and b^3 from p and q. By the way, the element δ above is closely related, but not identical to, the element with the same name in Problem 2.) (4 points)

3. Explain why Cardano's formula shows that cubic equations can be solved by radicals by constructing a root tower for the equation. (1 point)

Due date: Wednesday, May 6, 2015. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy the questions to your solution or to submit this sheet with your solution.