University at Buffalo
Yorck Sommerhäuser

Fall Semester 2014
MTH 619: Sheet 9

Algebra I

Problem 1: Suppose that G is a nonabelian group of order $4p$, where $p \geq 5$ is a prime.

1. Suppose that the 2-Sylow subgroups of G are isomorphic to $\mathbb{Z}_4$. Show that G is isomorphic to a semidirect product

$$G \cong \mathbb{Z}_4 \ltimes_{\varphi} \mathbb{Z}_p$$

where $\varphi : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism that is not trivial (i.e., not constantly equal to the unit element). (2 points)

2. Suppose further that the kernel of φ has order 2. Show that G contains elements x and y that generate G and satisfy

$$x^{2p} = 1 \qquad x^p = y^2 \qquad xy = yx^{-1}$$

but not $x^p = 1$. (Remark: If the assertions made after Problem 3 on Sheet 8 are correct, then G must therefore be isomorphic to the dicyclic group of order $4p$.) (3 points)

Problem 2: Suppose that p is a prime that satisfies $p \equiv 1 \pmod 4$.

1. Show that there are exactly two injective homomorphisms φ and ψ from $\mathbb{Z}_4$ to $\mathrm{Aut}(\mathbb{Z}_p)$. (1 point)

2. Show that the corresponding semidirect products

$$\mathbb{Z}_4 \ltimes_{\varphi} \mathbb{Z}_p \qquad \text{and} \qquad \mathbb{Z}_4 \ltimes_{\psi} \mathbb{Z}_p$$

are isomorphic. (3 points)

3. Show that these two isomorphic semidirect products are not isomorphic to the semidirect product considered in the second part of the preceding problem. (3 points)

(Hint: The reason is not that we have used a different homomorphism from $\mathbb{Z}_4$ to $\mathrm{Aut}(\mathbb{Z}_p)$ there; as we just saw, different homomorphisms can lead to isomorphic semidirect products.)

Problem 3: We denote by S_4 the symmetric group on four letters.

1. On a separate sheet of paper, draw the subgroup lattice of S_4. Explain why your picture contains all subgroups. (2 points)
2. Find the center and the derived group of S_4. (2 points)
3. Find the Sylow subgroups of S_4. (1 point)
4. Find the normalizers and the cores of the Sylow subgroups of S_4.(1 point)

(Prove all your assertions in full detail.)

Problem 4: Determine under which conditions on n the dihedral group D_{2n} of order $2n$ is nilpotent. (Prove all your assertions in full detail.) (2 points)

Due date: Wednesday, November 19, 2014. Please work in teams of two students. Every submitted solution should carry exactly two names, and each team member should have written up two of the problems. Please write your solution on letter-sized paper. It is not necessary to submit this sheet with your solution.