University at Buffalo **Fall Semester 2014**
Yorck Sommerhäuser **MTH 619: Sheet 8**

Algebra I

Problem 1: Suppose that X and H are finite groups and that $\varphi : X \to \mathrm{Aut}(H)$ is a group homomorphism. We know (page 34) that

$$\varepsilon_X : X \to X \ltimes_\varphi H, x \mapsto (x, 1) \qquad \text{and} \qquad \varepsilon_H : H \to X \ltimes_\varphi H, h \mapsto (1, h)$$

are group homomorphisms into the associated semidirect product.

1. Show that $\varepsilon_X(x)^{-1}\varepsilon_H(h)\varepsilon_X(x) = \varepsilon_H(h^x)$. (1 point)

2. Suppose that $f_X : X \to G$ and $f_H : H \to G$ are group homomorphisms into another group G that satisfy

   $$f_X(x)^{-1} f_H(h) f_X(x) = f_H(h^x)$$

   Show that there is a unique group homomorphism $f : X \ltimes_\varphi H \to G$ satisfying $f \circ \varepsilon_X = f_X$ and $f \circ \varepsilon_H = f_H$. (2 points)

(Remark: This is the so-called universal property of the semidirect product.)

Problem 2: Let a be a generator of the cyclic group C_4 and b be a generator of the cyclic group C_3. There is a unique group homomorphism $\varphi : C_4 \to \mathrm{Aut}(C_3)$ that maps a to the inversion mapping. In the associated semidirect product $C_4 \ltimes_\varphi C_3$ of order 12, we consider the elements $u := \varepsilon_{C_4}(a)$ and $v := \varepsilon_{C_3}(b)$.

1. Show that

   $$u^4 = 1 \qquad v^3 = 1 \qquad vu = uv^{-1}$$

2. Use Problem 1 to show that if G is another group that contains elements w and t that satisfy

   $$w^4 = 1 \qquad t^3 = 1 \qquad tw = wt^{-1}$$

   then there exists a unique group homomorphism $f : C_4 \ltimes_\varphi C_3 \to G$ such that $f(u) = w$ and $f(v) = t$. (4 points)

(Remark: Make sure that you prove the uniqueness. In case that this universal property is satisfied, we say that the relations $u^4 = 1$, $v^3 = 1$, $vu = uv^{-1}$ are defining relations.)

Problem 3: Suppose that G is a group.

1. If G is generated by elements w and t that satisfy
$$w^4 = 1 \qquad t^3 = 1 \qquad tw = wt^{-1}$$
define $x := w^2 t$ and $y := w$. Show that G is generated by x and y and that
$$x^6 = 1 \qquad x^3 = y^2 \qquad xy = yx^{-1}$$

2. Conversely, if G is generated by elements x and y that satisfy
$$x^6 = 1 \qquad x^3 = y^2 \qquad xy = yx^{-1}$$
define $t := y^2 x$ and $w := y$. Show that G is generated by t and w and that
$$w^4 = 1 \qquad t^3 = 1 \qquad tw = wt^{-1}$$

(Remark: In general, a group that is generated by elements x and y that satisfy
$$x^{2n} = 1 \qquad x^n = y^2 \qquad xy = yx^{-1}$$
is called a dicyclic group if these relations are defining. Such a group has order $4n$. In the case where n is a power of 2, such a group is called a generalized quaternion group. Our case is the case $n = 3$. The quaternion group Q_8 is the case $n = 2$.) (4 points)

Problem 4: Suppose that G is a nonabelian group of order $4p$, where $p \geq 5$ is a prime.

1. Show that there is a unique p-Sylow subgroup. (2 points)
2. Suppose that the 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Show that G is isomorphic to a semidirect product
$$G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes_\varphi \mathbb{Z}_p$$
where $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism. (2 points)
3. Show that φ is not injective and not trivial (i.e., not constantly equal to the unit element). Therefore, its kernel has order 2. (1 point)
4. Find a nontrivial element in the center of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes_\varphi \mathbb{Z}_p$. (1 point)
5. Show that $G \cong D_{4p}$. (3 points)

(Remark: The case where the 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}_4$ will be treated on the next exercise sheet.)

Due date: Wednesday, November 12, 2014. Please work in teams of two students. Every submitted solution should carry exactly two names, and each team member should have written up two of the problems. Please write your solution on letter-sized paper. It is not necessary to submit this sheet with your solution.