University at Buffalo **Fall Semester 2014**
Yorck Sommerhäuser **MTH 619: Sheet 7**

Algebra I

Problem 1: Suppose that G is a nonabelian group of order 8.

1. Show that G contains an element x of order 4. (1 point)
2. Show that $\langle x \rangle$ is normal. (1 point)
3. Show that any element $y \notin \langle x \rangle$ satisfies $y^2 \in \langle x \rangle$ and $y^{-1}xy = x^{-1}$. (1 point)
4. Show that $G \cong Q_8$ or $G \cong D_8$. (3 points)

Problem 2: Suppose that p and q are primes and that p divides $q-1$. For two monomorphisms $\varphi : C_p \to \operatorname{Aut}(C_q)$ and $\psi : C_p \to \operatorname{Aut}(C_q)$, show that the semidirect products

$$C_p \ltimes_\varphi C_q \qquad \text{and} \qquad C_p \ltimes_\psi C_q$$

are isomorphic. (6 points)

Problem 3: We denote by A_4 the alternating group on four letters.

1. On a separate sheet of paper, draw the subgroup lattice of A_4. Explain why your picture contains all subgroups. (2 points)
2. Find the center and the derived group of A_4. (2 points)
3. Find the Sylow subgroups of A_4. (1 point)
4. Find the normalizers and the cores of the Sylow subgroups of A_4.(1 point)

Problem 4: If a group G is acting on two set Ω and Ω', a map $\varphi : \Omega \to \Omega'$ is called G-equivariant if

$$(xg)^\varphi = x^\varphi g$$

for all $x \in \Omega$ and all $g \in G$. Now consider the case $G = A_4$, $\Omega = \{1, 2, 3, 4\}$, and let Ω' be the set of 3-Sylow subgroups of A_4, with the conjugation action. Construct a bijective G-equivariant map between Ω and Ω'. (2 points)

Due date: Wednesday, October 29, 2014. Please work in teams of two students. Every submitted solution should carry exactly two names, and each team member should have written up two of the problems. Please write your solution on letter-sized paper. It is not necessary to submit this sheet with your solution.