Algebra I

Problem 1: Suppose that H is a subgroup of index 2 in the group G. Show that H is normal in G. (2 points)

Problem 2:

1. Show that the identity and the inversion $g \mapsto g^{-1}$ are the the only automorphisms of C_3. Conclude that $\mathrm{Aut}(C_3) \cong C_2$. (1 point)
2. Let $\varphi : C_2 \to \mathrm{Aut}(C_3)$ be the isomorphism just constructed, and consider the associated semidirect product $C_2 \ltimes C_3$. Show that every nonabelian group G of order 6 is isomorphic to this semidirect product. (6 points)
3. Find an explicit isomorphism between the symmetric group S_3 and the dihedral group D_6 by listing which element is mapped to which element. Use the preceding part to explain why your map is an isomorphism. (2 points)

Problem 3: Suppose that H_1, H_2, H_3, and H_4 are subgroups of a group G. Without referencing the external direct product, show that G is the internal direct product of H_1, H_2, H_3, and H_4 if and only if

1. $U_1 := H_1 H_2$ is the internal direct product of H_1 and H_2.
2. $U_2 := H_3 H_4$ is the internal direct product of H_3 and H_4.
3. G is the internal direct product of U_1 and U_2.

Afterwards, formulate, but do not prove, a generalization of this problem to an arbitrary finite number of subgroups.
(Remarks: Consider carefully what you have to prove here. The more difficult part is the 'if' part. The statement can be summarized by saying that

$$H_1 \times H_2 \times H_3 \times H_4 \cong (H_1 \times H_2) \times (H_3 \times H_4)$$

Recall that direct products are discussed in Section 1.6 of the textbook, although the handed out pages from Jacobson also treat direct products. For the generalization, every correct answer will be accepted, but the better answers are those that are more general. The results of this exercise are frequently used without mention, for example in Theorem 1.7.2 of the textbook.) (6 points)

Problem 4:

1. Find a composition series for the alternating group A_4. (2 points)
2. Find a composition series for the symmetric group S_4. (1 point)

Due date: Monday, October 6, 2014. Please work in teams of two students. Every submitted solution should carry exactly two names, and each team member should have written up two of the problems. Please write your solution on letter-sized paper. It is not necessary to submit this sheet with your solution.