University at Buffalo
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Fall Semester 2014
MTH 619: Sheet 10

Algebra I

Problem 1: Suppose that $G_1, \ldots, G_n$ are subgroups of a group G. Generalizing the definitions made on page 25 of the textbook, we define

$$[G_1, \ldots, G_n] := [[\ldots [G_1, \ldots, G_{n-2}], G_{n-1}], G_n]$$

Moreover, for elements $g_1, \ldots, g_n \in G$ we define

$$[g_1, \ldots, g_n] := [[\ldots [g_1, \ldots, g_{n-2}], g_{n-1}], g_n]$$

If $G_1, \ldots, G_n$ are normal, show that

$$[G_1, \ldots, G_n] := \langle [g_1, \ldots, g_n] \mid g_1 \in G_1, \ldots, g_n \in G_n \rangle$$

(Hint: Use induction on n and the commutator formulas in 1.5.4 as well as their consequence $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$.) (8 points)

Problem 2: If $G = S_5$ is the symmetric group on five letters, consider the subgroups

$$G_1 = \langle (1,2) \rangle \qquad G_2 = \langle (2,3) \rangle \qquad G_3 = \langle (3,4) \rangle \qquad G_4 = \langle (2,5) \rangle$$

of order 2.

1. Show that $[g_1, g_2, g_3, g_4] = 1$ for all $g_1 \in G_1$, $g_2 \in G_2$, $g_3 \in G_3$, and $g_4 \in G_4$. (2 points)
2. Show that $[G_1, G_2, G_3, G_4] \neq \{1\}$. (2 points)

Problem 3: Find a finite nilpotent group for which not every subgroup that appears in the ascending central series appears also in the descending central series, and conversely not every subgroup that appears in the descending central series appears also in the ascending central series.
(Hint: In fact, most nilpotent groups will work. It is easiest to consider groups of nilpotent class 2.) (4 points)

Problem 4: Consider any field K, finite or infinite. For the subgroup

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in K \right\}$$

of $\mathrm{GL}(3, K)$, find the ascending central series and the descending central series, and decide whether the group is nilpotent. (4 points)
(Remark: This groups appears in many different areas of mathematics and has different names there. In some areas, it is called the Heisenberg group.)

Due date: Monday, December 1, 2014. Please work in teams of two students. Every submitted solution should carry exactly two names, and each team member should have written up two of the problems. Please write your solution on letter-sized paper. It is not necessary to submit this sheet with your solution.