University at Buffalo **Fall Semester 2014**
Yorck Sommerhäuser **MTH 619: Sheet 1**

Algebra I

Problem 1: For a natural number $n \geq 2$, we consider the complex number $\zeta := e^{2\pi i/n}$, and denote the multiplication by ζ by

$$r : \mathbb{C} \to \mathbb{C},\ z \mapsto \zeta z$$

In addition, we denote complex conjugation by

$$s : \mathbb{C} \to \mathbb{C},\ z \mapsto \bar{z}$$

1. Inside the group $\mathrm{Sym}(\mathbb{C})$ of all bijective mappings from $\mathbb{C}$ to itself with identity element $1 = \mathrm{id}_{\mathbb{C}}$, show that

$$r^n = 1 \qquad s^2 = 1 \qquad rs = sr^{-1}$$

(1 point)

2. Prove that the mappings $1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}$ are all distinct. (2 points)

3. Show that the subgroup $D_{2n} := \langle r, s \rangle$ of $\mathrm{Sym}(\mathbb{C})$ generated by r and s contains $2n$ elements. It is called the dihedral group of order $2n$.(2 points)

4. For $n = 5$, draw a rough sketch of the points $1, \zeta, \zeta^2, \zeta^3, \zeta^4$ in the complex plane. (1 point)

Problem 2: Suppose that G is a finite group that is generated by two distinct elements that both have order 2. Show that there exists a natural number $n \geq 2$ such that $G \cong D_{2n}$. (4 points)

Problem 3: The complex-valued matrices

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \text{and} \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are invertible and therefore define bijective (linear) mappings from $\mathbb{C}^2$ to itself. This also holds for the matrices $K := IJ$ and $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Show that the matrices $E, I, J, K, -E, -I, -J, -K$ are all distinct. (2 points)

2. Show that the subgroup $Q_8 := \langle I, J \rangle$ of $\mathrm{Sym}(\mathbb{C}^2)$ generated by I and J contains 8 elements. It is called the quaternion group. (2 points)

3. Make a group table for Q_8. (2 points)

Problem 4: The groups Q_8 and D_8 both contain eight elements. Decide whether the groups are isomorphic, and justify your decision by a detailed proof. (4 points)

Due date: Monday, September 8, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.