Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 3300: Sheet 8**

Set Theory

Problem 1: Suppose that X is a well-ordered set. Show that the lexicographical ordering on $X \times X$ introduced in Problem 4 on Sheet 6 is a well-ordering. (25 points)

Problem 2: For every $n \in \omega$, we consider the set

$$X_n := \left\{ \frac{m}{2^n} \mid m \in \omega \right\}$$

1. Show that X_n is well-ordered. (3 points)
2. Show that $X_n \subset X_{n+1}$. (1 point)
3. Show that $\bigcup_{n=0}^{\infty} X_n$ is not well-ordered. (16 points)
4. Decide whether $\{X_n \mid n \in \omega\}$ is a chain (with respect to continuation) in the sense described on page 67/68 of the textbook. (5 points)

(Remark: Although we have not constructed the rationals in class, you can use the elementary properties of rational numbers in this problem.)

Problem 3: Suppose that W is a well-ordered set, and that f is a sequence function in a set X. Suppose that U and V are functions from W to X satisfying $U(a) = f(U^a)$ and $V(a) = f(V^a)$ for all $a \in W$. Show that $U = V$. (25 points)

(Remark: This is the uniqueness part of the transfinite recursion theorem stated on page 70 of the textbook. The definition of a sequence function is also stated there.)

Problem 4: Show that well-ordering theorem implies the axiom of choice.
(Hint: For a collection $\mathcal{C}$ of nonempty sets, construct a choice function by choosing a certain smallest element.) (25 points)

Due date: Monday, March 27, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.