Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 3300: Sheet 7**

Set Theory

Problem 1: Zermelo's postulate states that, for a collection $\mathcal{C}$ of pairwise disjoint, nonempty sets, there exists a set A such that, for every $C \in \mathcal{C}$, the intersection $A \cap C$ contains exactly one element. Deduce Zermelo's postulate from the axiom of choice. (25 points)

Problem 2: Show that the axiom of choice can be deduced from Zorn's lemma. (Hint: Look at the hint on page 65 of the textbook.) (25 points)

Problem 3: A set X together with two mappings

$$\sqcap : X \times X \to X,\ (x, y) \mapsto x \sqcap y$$

and

$$\sqcup : X \times X \to X,\ (x, y) \mapsto x \sqcup y$$

is called a lattice if both of them are associative, i.e., we have

$$(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z) \qquad (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$$

both of them are commutative, i.e., we have

$$x \sqcap y = y \sqcap x \qquad x \sqcup y = y \sqcup x$$

and they satisfy the so-called absorption laws

$$x \sqcap (x \sqcup y) = x \qquad x \sqcup (x \sqcap y) = x$$

1. Show that by defining the relation $x \leq y :\Leftrightarrow x \sqcap y = x$, the set X becomes a partially ordered set. (15 points)

2. Show that, with respect to this ordering, $x \sqcup y$ is the supremum of the set $\{x, y\}$ and that $x \sqcap y$ is the infimum of the set $\{x, y\}$. (10 points)

Problem 4: Conversely, suppose that X is a partially ordered set with the property that, for all $x, y \in X$, the set $\{x, y\}$ has a supremum and an infimum. If the supremum of this set is denoted by $x \sqcup y$ and the infimum is denoted by $x \sqcap y$, show that X is a lattice with respect to these operations. (25 points)

Due date: Monday, March 20, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.