Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 3300: Sheet 6**

Set Theory

Problem 1: Suppose that E and F are finite sets. Show that the Cartesian product $E \times F$ is finite and that $\sharp(E \times F) = \sharp(E)\sharp(F)$. (25 points)
(Hint: Consider first the case where $E = m$ and $F = n$ are natural numbers, and show that the map

$$m \times n \to m \cdot n, \ (i, j) \mapsto n \cdot i + j$$

is bijective. For this, use Problem 2 on Sheet 5.)

Problem 2: Suppose that E is a finite set. Show that the power set $\mathcal{P}(E)$ is finite and that $\sharp\mathcal{P}(E) = 2^n$ if $\sharp E = n$. (25 points)
(Hint: Consider the case $E = n \in \omega$ first. Use induction and the definition of exponentiation on page 51 of the textbook and also discussed in the midterm.)

Problem 3:

1. Suppose that $<$ is a transitive relation on a set X with the property that, for all $x, y \in X$, we never have $x < y$ and $y < x$ simultaneously. In other words, if consider the relation as a set of pairs and write $R = <$, we have $R \cap R^{-1} = \emptyset$. If we define

$$x \leq y := x < y \ \text{ or } \ x = y$$

show that $\leq$ is a partial order on X. (13 points)

2. Conversely, if $\leq$ is a partial order on X, we define

$$x < y := x \leq y \ \text{ and } \ x \neq y$$

Show that $<$ is a transitive relation on X with the property that we never have $x < y$ and $y < x$ simultaneously. (12 points)

Problem 4: Suppose that X is a totally ordered set. On the set $X \times X$, we define the relation

$$(x, y) < (x', y') :\Leftrightarrow x < x' \ \text{ or } \ (x = x' \ \text{ and } \ y < y')$$

Show that this relation defines a total order on $X \times X$. (25 points)
(Hint: This order is called the lexicographical order on $X \times X$. The symbols $<$ and $\leq$ are related here as indicated in the previous problem.)

Due date: Monday, March 13, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.