

Set Theory

Problem 1: Recall the symmetric difference of two sets from Problem 4 on Sheet 1.

1. Show that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$. (12 points)
2. Show that $A \triangle \emptyset = A$. (3 points)

(Remark: This shows that the power set $\mathcal{P}(X)$ of a set X is a ring with symmetric difference as addition and intersection as multiplication. The neutral element for addition is $\emptyset$ and the neutral element for multiplication is X . This ring is a special case of a so-called Boolean ring.)

Problem 2: Suppose that X is a set and that $R \subset X \times X$ is a relation on X . We define the inverse relation

$$R^{-1} := \{(x, y) \in X \times X \mid (y, x) \in R\}$$

Show that R^{-1} is transitive if R is transitive. (15 points)

Problem 3:

1. Suppose that X is a set and that $\mathcal{C} \subset \mathcal{P}(X \times X)$ is a set of transitive relations on X . Show that $I := \bigcap \mathcal{C}$ is transitive. (15 points)
2. Suppose that $R \subset X \times X$ is a relation, and let

$$\mathcal{C} = \{S \subset X \times X \mid R \subset S \text{ and } S \text{ is transitive}\}$$

Then $I := \bigcap \mathcal{C}$ is called the transitive closure of R . Show that I is an equivalence relation if R is reflexive and symmetric. (15 points)

Problem 4: We have not yet defined what a number is and when a set is called finite. However, using your ‘naive’ understanding of these concepts, show that the power set of a finite set with n elements contains 2^n elements.

(Hint: Use mathematical induction. For the inductive step, fix an element of the set and consider the subsets that do not contain this element on the one hand and those subsets that do contain this element on the other hand.) (40 points)

Due date: Monday, January 30, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.

Change of syllabus: The midterm exam will take place on Friday, February 17, not on Wednesday, February 15.