Memorial University of Newfoundland **Fall Semester 2025**
Yorck Sommerhäuser **MATH 6333: Sheet 6**

Representation Theory

Problem 1: Consider the partition $\lambda = (2,2)$ of 4 as well as the Young tableaux

$$\mathbf{t_1} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \text{and} \qquad \mathbf{t_2} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

1. Write out the associated polytabloids $\mathbf{e_{t_1}}$ and $\mathbf{e_{t_2}}$ explicitly in terms of tabloids. (5 points)
2. Show that $\mathbf{e_{t_1}}$ and $\mathbf{e_{t_2}}$ form a basis of the Specht module S^λ. (20 points)

(Hint: Consider Problem 2 simultaneously.)

Problem 2: The elements $g_1 = \text{id}$, $g_2 = (1,2)$, $g_3 = (1,2)(3,4)$, $g_4 = (1,2,3)$, and $g_5 = (1,2,3,4)$ are representatives for the five conjugacy classes of S_4.

1. Find the matrix representations of the action of these five elements on S^λ with respect to the basis $\mathbf{e_{t_1}}$ and $\mathbf{e_{t_2}}$ from Problem 1. (20 points)
2. Decide which character in the table in Problem 4 on Sheet 3 corresponds to S^λ. (5 points)

Problem 3: Continuing the preceding two problems, we denote the orthogonal complement of S^λ with respect to the bilinear form described in Equation (2.3) on page 64 in the textbook by $(S^\lambda)^\perp$. Show that $(S^\lambda)^\perp$ is isomorphic to the direct sum of the trivial representation of S_4 and the Specht module S^μ, where μ is the partition $\mu = (3,1)$. (25 points)

(Hint: Recall from Example 2.3.8 in the textbook that S^μ is related to the defining representation.)

Problem 4: The group S_3 can be viewed as the subgroup of those permutations of S_4 that keep the letter 4 fixed. For $\lambda = (2,2)$ as above, decide whether the restriction of S^λ to S_3 is reducible or irreducible. In both cases, determine which irreducible S_3-modules appear in the restriction. (25 points)

Due date: Monday, November 3, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.