Memorial University of Newfoundland **Fall Semester 2025**
Yorck Sommerhäuser **MATH 6333: Sheet 5**

Representation Theory

Problem 1: Suppose that G is a finite group, that H is a subgroup of G of index $m := [G : H]$, and that U is a subgroup of H of index $k := [H : U]$. Suppose that $\tilde{\rho}\colon U \to \mathrm{GL}(n, K)$ is a matrix representation of U, and consider the induced representation $\tilde{\sigma} := \tilde{\rho} \uparrow_U^H \colon H \to \mathrm{GL}(nk, K)$ as well the induced representation $\tilde{\tau} := \tilde{\sigma} \uparrow_H^G \colon G \to \mathrm{GL}(nkm, K)$. Show that $\tilde{\tau} = \tilde{\rho} \uparrow_U^G$ is also the induced representation from U to G in one step.

(Hint: The definition of an induced representation can be found in Definition 1.12.2 in the textbook.) (25 points)

Problem 2: The dihedral group D_8 of order 8 contains the eight distinct elements $e, r, r^2, r^3, s, sr, sr^2, sr^3$, where

$$r^4 = e \qquad s^2 = e \qquad sr = r^{-1}s$$

Show that there are exactly four nonisomorphic one-dimensional representations

$$\omega_j \colon D_8 \to \mathbb{C}^\times$$

for $j = 1, 2, 3, 4$, and make a table that contains the values $\omega_j(r)$ and $\omega_j(s)$. (25 points)

Problem 3: The cyclic group $C_4 = \{e, r, r^2, r^3\}$ is a subgroup of D_8. It has four one-dimensional representations $\sigma_j \colon C_4 \to \mathbb{C}^\times$, for $j = 1, 2, 3, 4$, determined by

$$\sigma_1(r) = 1 \qquad \sigma_2(r) = i \qquad \sigma_3(r) = -1 \qquad \sigma_4(r) = -i$$

where $i \in \mathbb{C}$ denotes the imaginary unit. We consider the induced representations

$$\rho_j := \sigma_j \uparrow_{C_4}^{D_8} \colon D_8 \to \mathrm{GL}(2, \mathbb{C})$$

1. Show that ρ_1 and ρ_3 are reducible. (10 points)
2. Show that ρ_2 and ρ_4 are irreducible and isomorphic. (15 points)

Problem 4:

1. Describe the conjugacy classes of D_8 by specifying which of the eight elements they contain. Beginning with the conjugacy class of the unit element, list them so that their orders are weakly increasing. (10 points)

2. Make a character table for D_8. Beginning with the trivial representation, list the characters so that their degrees are weakly increasing. (15 points)

Due date: Monday, October 27, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.