Memorial University of Newfoundland **Fall Semester 2025**
Yorck Sommerhäuser **MATH 6333: Sheet 4**

Representation Theory

Problem 1: Suppose that G is a group, that V is a G-module, and that

$$V \times V \to K,\ (v, w) \mapsto \langle v, w \rangle$$

is a symmetric bilinear form with values in the base field K. We assume that the bilinear form is invariant in the sense that $\langle g.v, g.w \rangle = \langle v, w \rangle$ for all $g \in G$ and all $v, w \in V$.

In this situation, show that, for a G-submodule W of V, the orthogonal complement

$$W^{\perp} := \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$$

is again a G-submodule.

(Remark: A very similar statement, with a very similar proof, holds for inner products over the complex numbers.) (25 points)

Problem 2: Suppose that G is a finite group and that V is a finite-dimensional G-module over the real numbers $\mathbb{R}$. Show that there exists a G-invariant inner product on V, i.e., a G-invariant symmetric bilinear form that is positive definite.

(Hint: First argue that there is some inner product and then average it to make it G-invariant. A very similar statement, with a very similar proof, holds for inner products over the complex numbers.) (25 points)

Problem 3: Suppose that G is a finite group and that H is a subgroup of G. Suppose that $\tilde{\rho}\colon H \to \mathrm{GL}(n, K)$ is a matrix representation of H. For two transversals $t_1, \ldots, t_k$ and $s_1, \ldots, s_k$ of H in G, consider the induced representations

$$\tilde{\rho}'(g) := (\tilde{\rho}(t_i^{-1} g t_j))_{i,j \leq k} \qquad \text{and} \qquad \tilde{\rho}''(g) := (\tilde{\rho}(s_i^{-1} g s_j))_{i,j \leq k}$$

where $\tilde{\rho}(g) := 0$ if $g \notin H$. Find an explicit matrix $T \in \mathrm{GL}(nk, K)$ such that

$$\tilde{\rho}''(g) = T \tilde{\rho}'(g) T^{-1}$$

for all $g \in G$.

(Hint: The definition of an induced representation can be found in Definition 1.12.2 in the textbook. First assume that $t_i H = s_i H$ for all $i = 1, \ldots, k$, then discuss the general case.) (25 points)

Problem 4: The symmetric group S_4 acts on the set $S = \{1, 2, 3, 4\}$. The associated permutation representation is the S_4-module $V := K[S]$. We denote its character by χ.

1. Describe the conjugacy classes C_1, C_2, C_3, C_4, and C_5 of S_4. List them in a way so that they contain 1, 6, 3, 8, and 6 elements, respectively, as in Problem 4 on Sheet 3. (10 points)

2. Assuming that S_4 has the character table found in Problem 4 on Sheet 3, write χ as a linear combination of χ_1, χ_2, χ_3, χ_4, and χ_5. (15 points)

Due date: Wednesday, October 15, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.