Memorial University of Newfoundland **Fall Semester 2025**
Yorck Sommerhäuser **MATH 6333: Sheet 3**

Representation Theory

Problem 1: Suppose that $\rho\colon G \to \mathrm{GL}(V)$ is a representation of the finite group G in the finite-dimensional vector space V over the algebraically closed field K of characteristic zero. Suppose that the only G-module homomorphisms $\theta\colon V \to V$ are the scalar multiples of the identity, i.e., those of the form $\theta = \lambda\,\mathrm{id}_V$ for $\lambda \in K$. Show that V is irreducible.

(Hint: This statement can be viewed as a converse to Schur's lemma.) (25 points)

Problem 2: Suppose that $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$ is a complex matrix representation of the finite group G with character χ. Suppose that, for an element $g \in G$, we have $\chi(g) = n$. Show that $\rho(g) = I_n$, the $n \times n$-identity matrix.

(Hint: How large can the sum of n roots of unity possibly be?) (25 points)

Problem 3: The symmetric group S_3 in three letters acts on the set $S = \{1, 2, 3\}$. The associated permutation representation is the S_3-module $V = K[S]$. We denote the basis vectors corresponding to the elements of the set S by v_1, v_2, v_3 respectively.

1. Show that

$$W := \{\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \mid \lambda_1 + \lambda_2 + \lambda_3 = 0\}$$

is a S_3-submodule. (5 points)

2. Show that W is irreducible if $\mathrm{char}(K) \neq 3$. (20 points)

Problem 4: Suppose that G is a group containing 24 elements, partitioned into five conjugacy classes C_1, C_2, C_3, C_4, and C_5. We assume that C_1 consists only of the unit element, while the remaining conjugacy classes contain 6, 3, 8, and 6 elements, respectively. Let χ_1, χ_2, χ_3, χ_4, and χ_5 be the distinct irreducible characters, where χ_1 is the character of the trivial representation. Complete the following character table:

	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5					

(25 points)

Due date: Monday, October 6, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.