Memorial University of Newfoundland **Fall Semester 2025**
Yorck Sommerhäuser **MATH 6333: Sheet 2**

Representation Theory

Problem 1: Suppose that K is a field and that G is a group. On the free vector space $K[G]$ with basis G, we introduce a multiplication as follows: For two elements $a = \sum_{g \in G} \alpha_g g$ and $b = \sum_{g \in G} \beta_g g$, where $\alpha_g, \beta_g \in K$ and all but finitely many are zero, we define

$$ab := \sum_{g,h \in G} \alpha_g \beta_h (gh)$$

Show that this multiplication makes $K[G]$ into an algebra, called the group algebra of G.

(Hint: You need to show the associative law, i.e., $(ab)c = a(bc)$, two distributive laws, the existence of a unit element for the multiplication introduced above, as well as the equation

$$(\lambda a)b = a(\lambda b) = \lambda(ab)$$

for all $a, b \in K[G]$ and $\lambda \in K$.) (25 points)

Problem 2: Suppose that K is a field and that G is a group.

1. For a representation $\rho\colon K[G] \to \mathrm{End}_K(V)$, show that the restriction of ρ to the basis G yields a representation of G.
2. Conversely, suppose that $\rho'\colon G \to \mathrm{GL}(V)$ is a representation of G. Show that there is a representation $\rho\colon K[G] \to \mathrm{End}_K(V)$ with the property that $\rho(g) = \rho'(g)$ for all $g \in G$.

(Hint: By definition, a representation of an algebra A is an algebra homomorphism $\rho\colon A \to \mathrm{End}_K(V)$, where V is a vector space over the same field K as A. In the first part, you need to show in particular that $\rho(g)$ is bijective for all $g \in G$.) (25 points)

Problem 3: Show that the set

$$\left\{\lambda \sum_{g\in S_3} \operatorname{sgn}(g)g \mid \lambda \in K\right\}$$

is a (two-sided) ideal in $K[S_3]$.

(Hint: By definition, a (two-sided) ideal in an algebra A is a subgroup I of the additive group with the property that $ax \in I$ and $xa \in I$ for all $a \in A$ and all $x \in I$. It is automatically a subspace, because $\lambda x = (\lambda e)x$ for all $\lambda \in K$ and all $x \in I$, where e denotes the unit element of A.) (25 points)

Problem 4: Suppose that $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional irreducible representation of the abelian group G over an algebraically closed field K. Show that V is one-dimensional. (25 points)

Due date: Monday, September 29, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.