Memorial University of Newfoundland **Fall Semester 2017**
Yorck Sommerhäuser **MATH 2051: Sheet 9**

Linear Algebra II

Problem 1: Prove that the following identity, called the polarisation identity, holds for vectors in any inner product space:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$$

Note that this identity shows that not only the inner product determines the norm, but also conversely the norm determines the inner product. Justify all your arguments with the help of the algebraic properties of inner products discussed at the (end of) Section 6.1. (25 points)

Problem 2: Let the space $C[0,1]$ of continuous functions on the unit interval have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)dx$$

With respect to this inner product, find the cosine of the angle between the vectors $\mathbf{p} = p(x) = x$ and $\mathbf{q} = q(x) = x^2$. (25 points)

Problem 3: For vectors in any inner product space V, recall that the Cauchy-Schwarz inequality states that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

Find an inner product space V and vectors $\mathbf{u}$ and $\mathbf{v}$ so that the Cauchy-Schwarz inequality, applied to this situation, implies that

$$(a\cos(\theta) + b\sin(\theta))^2 \leq a^2 + b^2$$

for all real numbers a, b, and θ. (25 points)

Problem 4: In the vector space R^3 with the standard Euclidean inner product, use the Gram-Schmidt process to transform the basis

$$\mathbf{u}_1 = (1, 0, 0), \qquad \mathbf{u}_2 = (3, 7, -2), \qquad \mathbf{u}_3 = (0, 4, 1)$$

into an orthonormal basis. (25 points)

Due date: Monday, November 20, 2017. Work in groups of three students. Write your solution on letter-sized paper, and write your names on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.