Memorial University of Newfoundland **Fall Semester 2017**
Yorck Sommerhäuser **MATH 2051: Sheet 4**

Linear Algebra II

Problem 1: In the vector space P_1 of polynomials of degree at most 1, we consider two bases $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$, where

$$\mathbf{p}_1 = 6 + 3x, \qquad \mathbf{p}_2 = 10 + 2x, \qquad \mathbf{q}_1 = 2, \qquad \mathbf{q}_2 = 3 + 2x.$$

(You do not need to show that these form indeed bases.)

1. Find the transition matrix $P_{B' \to B}$ from B' to B. (6 points)
2. Find the transition matrix $P_{B \to B'}$ from B to B'. (6 points)
3. For the polynomial $\mathbf{p} = -4+x$, find the coordinate vectors $(\mathbf{p})_B$ and $(\mathbf{p})_{B'}$ directly. (6 points)
4. Confirm Equations (11) and (12) on page 232 of the textbook by verifying explicitly that $(\mathbf{p})_{B'} = P_{B \to B'}(\mathbf{p})_B$ and $(\mathbf{p})_B = P_{B' \to B}(\mathbf{p})_{B'}$. (6 points)

Problem 2: Consider the basis $B' = \{(1,1,1), (1,1,0), (1,0,0)\}$ of R^3 and the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Find a basis B of R^3 so that $P = P_{B \to B'}$. (25 points)
(You do not need to show that B' is a basis, but you need to show that B is a basis.)

Problem 3: Prove the following important theorem, known as the universal property of a basis: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ are vectors in a vector space W, not necessarily distinct, then there exists a unique linear transformation $T : V \to W$ such that

$$T(\mathbf{v}_1) = \mathbf{w}_1, \qquad T(\mathbf{v}_2) = \mathbf{w}_2, \qquad \ldots \qquad T(\mathbf{v}_n) = \mathbf{w}_n.$$

(You need to show two things: existence and uniqueness). (26 points)

Problem 4: Consider the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where

$$\mathbf{v}_1 = (-2, 1) \qquad \text{and} \qquad \mathbf{v}_2 = (1, 3)$$

(You do not need to show that B is a basis.) By Problem 3, there is a unique linear transformation $T : R^2 \to R^3$ with the property that

$$T(\mathbf{v}_1) = (-1, 2, 0) \qquad \text{and} \qquad T(\mathbf{v}_2) = (0, -3, 5)$$

Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2, -3)$. (25 points)

Due date: Monday, October 16, 2017. Work in groups of three students. Write your solution on letter-sized paper, and write your names on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.