Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 8**

Lie Algebras

Problem 1: Suppose that K is an algebraically closed field of characteristic zero, and that V is a finite-dimensional vector space.

1. Show that the Lie algebra $\mathrm{gl}(V)$ is reductive. (15 points)
2. Show that $\mathrm{gl}(V)$ is not semisimple. (10 points)

(Remark: The definition of a reductive Lie algebra was given in Problem 4 on Sheet 7. In view of this definition, it follows from Weyl's theorem that a semisimple Lie algebra is reductive.)

Problem 2: Suppose that K is a field of characteristic zero, so that its prime field is isomorphic to the field $\mathbb{Q}$ of rational numbers. Let m be a nonnegative integer and V be a vector space over K with basis $v_0, v_1, \ldots, v_m$, so that $\dim(V) = m + 1$. Let $v_{-1} = v_{m+1} := 0$ and let H, X, and Y be the endomorphisms of V determined on the given basis by the formulas

1. $H(v_i) := (m - 2i)v_i$
2. $Y(v_i) := (i + 1)v_{i+1}$
3. $X(v_i) := (m - i + 1)v_{i-1}$

for $i = 0, \ldots, m$.

1. Show that
$$[H, X] = 2X \qquad [H, Y] = -2Y \qquad [X, Y] = H$$
(15 points)
2. Explain why these equations imply that the linear map
$$\phi\colon \mathrm{sl}(2, K) \to \mathrm{End}_K(V)$$
that is given on the basis x, y, h from Problem 4 on Sheet 1 as
$$\phi(x) = X \qquad \phi(y) = Y \qquad \phi(h) = H$$
defines a representation of $\mathrm{sl}(2, K)$. (2 points)
3. Show that this representation is irreducible. (8 points)

Problem 3: Suppose that K is a field of characteristic zero, that L is a Lie algebra over K, and that V is an L-module.

1. Show that the k-th exterior power $\bigwedge^k V$ has a unique L-module structure with the property that

$$x.(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge v_2 \wedge \ldots \wedge (x.v_i) \wedge \ldots \wedge v_k$$

   for $v_1, v_2, \ldots, v_k \in V$. (15 points)

2. Show that the k-th symmetric power $\bigvee^k V$ has a unique L-module structure with the property that

$$x.(v_1 \vee v_2 \vee \ldots \vee v_k) = \sum_{i=1}^{k} v_1 \vee v_2 \vee \ldots \vee (x.v_i) \vee \ldots \vee v_k$$

   for $v_1, v_2, \ldots, v_k \in V$. (10 points)

(Remark: Background on exterior and symmetric powers can be found in the book *Multilinear algebra* (2nd ed.) by Werner Greub. For this problem, all you need to know are the universal properties, given there on page 99f and page 210, respectively. This material is also discussed in several other books. A short account can be found in Appendix B of the book *Representation theory* by William Fulton and Joe Harris.)

Problem 4: Suppose that K is a field of characteristic zero and that W is the defining representation of $\mathrm{sl}(2, K)$, introduced in Problem 3 on Sheet 7. Show that the module V constructed in Problem 2 is isomorphic to the symmetric power $\bigvee^m W$. (25 points)

Due date: Thursday, March 22, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.