Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 7**

Lie Algebras

Problem 1: The usual vector space $\mathbb{R}^3$ over the real numbers $\mathbb{R}$ is a Lie algebra with respect to the cross product.

1. Show that $\mathbb{R}^3 \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}^3$, where the Lie bracket is defined via the same formula as for $\mathbb{R}^3$, but for vectors with complex components instead of real components. (7 points)
2. Show that $\mathbb{R}^3 \otimes \mathbb{C}$ is isomorphic to $\mathrm{sl}(2, \mathbb{C})$. (18 points)

(Remark: The Lie bracket on $\mathbb{R}^3$ was recalled on Problem 1 on Sheet 6. The Lie bracket on $\mathbb{R}^3 \otimes \mathbb{C}$ was defined in the second part of Problem 2 on Sheet 6.)

Problem 2: Suppose that L is a Lie algebra over a base field K and that V is a finite-dimensional L-module.

1. Show that V is isomorphic to V^* if and only if there is a nondegenerate bilinear form $\beta : V \times V \to K$ with the property that
$$\beta(x.v, w) = -\beta(v, x.w)$$
for all $v, w \in V$ and all $x \in L$. (8 points)
2. Show that V is irreducible if and only if the dual module V^* is irreducible. (8 points)
3. Suppose that K is algebraically closed of characteristic zero and that L is simple. Show that every associative bilinear form $\beta : L \times L \to K$ is proportional to the Killing form κ. (9 points)

(Remark: Note that, in the case where $V = L$ via the adjoint representation, the condition β in the first part is exactly the associativity condition. For the third part, use Schur's lemma. The third part should be compared with Problem 4 on Sheet 5.)

Problem 3: Suppose that K is a field whose characteristic is different from 2, and consider the Lie algebra $L = \mathrm{sl}(2, K)$. The representation

$$\phi : L \to \mathrm{End}(K^2)$$

that maps each matrix to the corresponding linear transformation of K^2 it describes is called the defining representation. Let $\beta : L \times L \to K$ be the corresponding bilinear form, given by

$$\beta(a, b) = \mathrm{Tr}(\phi(a)\phi(b)) = \mathrm{Tr}(ab)$$

for all $a, b \in L$. It follows from Problem 4 on Sheet 5 that β is nondegenerate.

1. For the basis given in Problem 4 on Sheet 1, find the dual basis with respect to β. (13 points)
2. Compute the Casimir element c_ϕ. (12 points)

Problem 4: Suppose that K is an algebraically closed field of characteristic zero. A finite-dimensional Lie algebra L over K is called reductive if its adjoint representation is completely reducible.

1. For a reductive Lie algebra L, show that $L = I \oplus J$, where I is a semisimple ideal and J is an abelian ideal. (An ideal is called semisimple if it is semisimple considered as a Lie algebra itself.) (9 points)
2. In this situation, show that $I = [L, L]$ and $J = Z(L)$. (8 points)
3. Show that $\mathrm{Rad}(L) = Z(L)$. (8 points)

Due date: Thursday, March 15, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.