Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 6**

Lie Algebras

Problem 1: The usual vector space $\mathbb{R}^3$ over the real numbers $\mathbb{R}$ is a Lie algebra with respect to the cross product.

1. Show that $\mathbb{R}^3$ is not isomorphic to $\mathrm{sl}(2, \mathbb{R})$.

   (Hint: Show that $\mathbb{R}^3$ does not contain an element that behaves like the element $h \in \mathrm{sl}(2, \mathbb{R})$ from Problem 4 on Sheet 1.) (24 points)

2. Speculate on the following question: During the midterm examination, you have shown that $\mathbb{R}^3$ is simple. In the first lecture, we have already seen the complete list of simple Lie algebras: There were four infinite series, which we have already introduced; they are in Section 1.2 of the textbook. In addition, there are five exceptional Lie algebras; none of them is 3-dimensional. How does $\mathbb{R}^3$ fit into this picture? (1 point)

(Remark: Recall that the cross product of $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is defined as $v \times w := (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$. You do not need to show that this defines a Lie algebra. In the second part, you do not need to prove anything; any reasonable answer is accepted.)

Problem 2: Suppose that L is a Lie algebra over the field K.

1. If A is an associative commutative algebra over K, show that $L \otimes A$ is a Lie algebra over K with respect to a Lie bracket that has the property

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

   (15 points)

2. Suppose that $P \supset K$ is a field extension of K. Show that $L \otimes P$ is a Lie algebra over P (not only over K) with respect to a Lie bracket that has the property

$$[x \otimes p, y \otimes q] = [x, y] \otimes pq$$

   (10 points)

(Remarks: For the first part, use the universal property of the tensor product to show that such a Lie bracket exists. For the second part, you can apply the first part, but as the comment in parentheses indicates, there is a small point left to do.)

Problem 3: Suppose that L is a Lie algebra and that V is a finite-dimensional L-module. Show that the following conditions on V are equivalent:

1. V is a direct sum of simple L-submodules.
2. For every L-submodule U of V, there is another L-submodule W so that $V = U \oplus W$.

(Remark: Such a module V is called semisimple, or completely reducible. In the proof of the second assertion from the first, you can actually use a sum of some of the simple submodules for the submodule W.) (25 points)

Problem 4: Suppose that L is a Lie algebra over the field K, and that V and W are L-modules.

1. Show that $\mathrm{Hom}_K(V, W)$ becomes an L-module with respect to the module structure
$$(x.f)(v) := x.f(v) - f(x.v)$$
for $f \in \mathrm{Hom}_K(V, W)$, $x \in L$, and $v \in V$. (13 points)
2. Show that the map $\Phi : W \otimes V^* \to \mathrm{Hom}_K(V, W)$ defined by
$$\Phi(w \otimes \varphi)(v) := \varphi(v)w$$
is a homomorphism of L-modules. (12 points)

(Remark: The module structures on the dual space and on the tensor product are described in Section 6.1 of the textbook. This section also contains a version of this problem with slightly different conventions. The map Φ is always injective, and its image are the linear maps of finite rank. The map is therefore bijective if either V or W is finite-dimensional. You do not need to prove these statements.)

Due date: Thursday, March 8, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.