Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 5**

Lie Algebras

Problem 1: Let K be a field of characteristic $p > 0$. Consider the $p \times p$-matrices

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

and $y = \operatorname{diag}(0, 1, 2, \dots, p-1)$ with entries in K.

1. Show that $[x, y] = x$. (5 points)
2. Show that $L := \operatorname{Span}(x, y)$ is a two-dimensional solvable Lie algebra. (10 points)
3. Show that x and y do not have a common eigenvector. (10 points)

(Remark: Note that this example shows that Lie's theorem does not hold in positive characteristic.)

Problem 2: We stay in the situation of Problem 1. Consider the space K^p as an abelian Lie algebra, i.e., as a Lie algebra with bracket identically equal to zero. Then every linear transformation is a derivation, so that $L \subset \operatorname{Der}(K^p)$. Let $\phi : L \to \operatorname{Der}(K^p)$ be the inclusion mapping, and let $M := K^p \oplus L$ be the (external) semidirect product described in Problem 1 on Sheet 3.

1. Show that M is solvable. (13 points)
2. Show that the derived algebra M' is not nilpotent. (12 points)

Problem 3: Suppose that V is a finite-dimensional vector space over an algebraically closed field K of characteristic zero, and that $L \subset \operatorname{gl}(V)$ is a solvable Lie subalgebra. Show that $\operatorname{Tr}(xy) = 0$ for all $x \in L'$ and $y \in L$, where L' denotes the derived algebra. (25 points)

(Remark: Note that this is the converse to Cartan's criterion for solvability.)

Problem 4: Suppose that K is a field whose characteristic is different from 2, and consider the Lie algebra $L = \mathrm{sl}(2, K)$.

1. Compute the fundamental matrix of the Killing form κ with respect to the basis given in Problem 4 on Sheet 1. (14 points)
2. Compute the determinant of this fundamental matrix and decide whether or not it is zero. (2 points)
3. Show that $\kappa(a, b) = 4\,\mathrm{Tr}(ab)$ for all $a, b \in L$. (9 points)

(Remark: The fundamental matrix was defined in Problem 1 on Sheet 1. Note that a bilinear form is nondegenerate if and only if its fundamental matrix has nonzero determinant.)

Due date: Tuesday, February 27, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.