Lie Algebras

Problem 1: Suppose that L is a Lie algebra, that $M \subset L$ is a Lie subalgebra, and that $I \subset L$ is an ideal. If $L = I \oplus M$, then L is called the (internal) semidirect product of I and M.

1. Show that $\phi : M \to \mathrm{Der}(I),\ x \mapsto \mathrm{ad}(x) \mid_I$ is a (well-defined) Lie algebra homomorphism. (10 points)
2. Show that L is isomorphic to the (external) semidirect product of I and M, constructed with the help of ϕ as described in Problem 1 on Sheet 3. (15 points)

Problem 2: Suppose that K is a field of characteristic zero. We partition the elements of the Lie algebra $\mathrm{so}(5, K)$ as described on page 3 of the textbook, i.e., in the form

$$\begin{pmatrix} 0 & b_1 & b_2 \\ -b_2^T & M & N \\ -b_1^T & P & -M^T \end{pmatrix}$$

where b_1 and b_2 are row vectors with two components and M, N, and P are 2×2-matrices subject to the requirement that N and P are skew-symmetric. As in the textbook, we consider the following basis: H_1 and H_2 are defined as

$$H_i := \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ii} & 0 \\ 0 & 0 & -E_{ii} \end{pmatrix}$$

B_1, B_2, C_1, and C_2 are defined via

$$B_i := \begin{pmatrix} 0 & 0 & e_i \\ -e_i^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C_i := \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ -e_i^T & 0 & 0 \end{pmatrix}$$

for $i = 1, 2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Furthermore, define

$$X := \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{12} & 0 \\ 0 & 0 & -E_{21} \end{pmatrix} \qquad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{21} & 0 \\ 0 & 0 & -E_{12} \end{pmatrix}$$

Finally, with the help of the 2×2-matrix $A = E_{12} - E_{21} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we define

$$R := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix} \qquad S := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A & 0 \end{pmatrix}$$

1. Use a computer algebra system of your choice to fill the blanks in the following table for the Lie brackets $[a, b]$:

$a \setminus b$	H_1	H_2	B_1	B_2	C_1	C_2	X	Y	R	S
H_1	*									
H_2	*	*								
B_1	*	*	*							
B_2	*	*	*	*						
C_1	*	*	*	*	*					
C_2	*	*	*	*	*	*				
X	*	*	*	*	*	*	*			
Y	*	*	*	*	*	*	*	*		
R	*	*	*	*	*	*	*	*	*	
S	*	*	*	*	*	*	*	*	*	*

   Do only fill the fields that do not contain a star, as the other fields follow follow from antisymmetry. Submit printouts of your computer algebra worksheets with your solution. (13 points)

2. Use this table and the results of Problem 3 on Sheet 2 to show that $\mathrm{so}(5, K)$, the Lie algebra of type B_2, is isomorphic to $\mathrm{sp}(4, K)$, the Lie algebra of type C_2. (12 points)

(Remark: Typical computer algebra systems are Maple, Mathematica, or Octave; there are many others. Octave is free software and does not require a license. It is a good idea to save your worksheets, as we might come back to this algebra for other computations.)

Problem 3: Consider the Lie algebra L from the first part of Problem 1 on Sheet 2.

1. Determine whether L is solvable. (12 points)
2. Determine whether L is nilpotent. (13 points)

Problem 4: Suppose that L is a finite-dimensional Lie algebra. Show that the following conditions on L are equivalent:

1. L is semisimple.
2. L does not contain a nonzero abelian ideal. (25 points)

Due date: Thursday, February 8, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.