Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 3**

Lie Algebras

Problem 1: Suppose that L and M are Lie algebras and that $\phi : M \to \mathrm{Der}(L)$ is a Lie algebra homomorphism. On the (external) direct sum $L \oplus M$, i.e., the set of all pairs with componentwise vector space structure, we introduce the Lie bracket

$$[(l, m), (l', m')] := ([l, l'] + \phi(m)(l') - \phi(m')(l), [m, m'])$$

1. Show that this indeed defines a Lie algebra structure on $L \oplus M$, called the (external) semidirect product. (20 points)
2. Show that $i : L \to L \oplus M,\ l \mapsto (l, 0)$ and $j : M \to L \oplus M,\ m \mapsto (0, m)$ are Lie algebra homomorphisms. (3 points)
3. Show that $i(L)$ is an ideal in $L \oplus M$. (2 points)

Problem 2: For a field K of characteristic zero, consider $L = \mathrm{sl}(2, K)$. With the notation of Problem 4 on Sheet 1, define $\sigma \in \mathrm{End}(L)$ as

$$\sigma := \exp(\mathrm{ad}(x)) \circ \exp(\mathrm{ad}(-y)) \circ \exp(\mathrm{ad}(x))$$

Show that

$$\sigma(x) = -y \qquad \sigma(y) = -x \qquad \sigma(h) = -h$$

(Hint: Instead of computing with the definition of σ, it is much easier to use the formula $\sigma(z) = szs^{-1}$ for a certain matrix s, as we saw in class. You can use this approach, but you need to compute s and explain briefly, without repeating the entire proof, why this approach works.) (25 points)

Problem 3: Let K be a field, and let $M := \mathfrak{n}(n, K)$ be the Lie subalgebra of $L := \mathrm{gl}(n, K)$ that consists of the strictly upper triangular matrices, i.e., upper triangular matrices with zeroes on the diagonal. Find the normalizer $N_L(M)$. (25 points)

Problem 4: Suppose that K is a field whose characteristic is not 2. According to Section 1.2 in the textbook, the Lie algebra of type B_1 is $L = \mathrm{so}(3, K)$.

1. Write down explicitly the 3×3-matrices that, according to the textbook, form a basis of L. (5 points)
2. Show that $\mathrm{so}(3, K) \cong \mathrm{sl}(2, K)$. (20 points)

Due date: Thursday, February 1, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.

Change of syllabus: Office hours on Thursday will now be from 4:00 pm to 5:00 pm.