Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 2**

Lie Algebras

Problem 1: A Lie algebra is called abelian if the Lie bracket of any two elements is equal to zero.

1. Show that
$$L := \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \,\middle|\, x, y \in K \right\}$$
is a two-dimensional nonabelian Lie subalgebra of $\mathrm{gl}(2, K)$, where K denotes the base field. (5 points)

2. Show that every two-dimensional nonabelian Lie algebra over K is isomorphic to L. (20 points)

Problem 2: Suppose that K is a field of characteristic zero. Show that for $L = \mathrm{sl}(n, K)$, we have $L = L'$, i.e., L is equal to its derived algebra. (25 points)

Problem 3: Suppose that K is a field of characteristic zero. For the Lie algebra $\mathrm{sp}(4, K)$, show that the basis given in Section 1.2 of the textbook can be enumerated as
$$h_1, h_2, x, y, n_1, n_2, n_3, p_1, p_2, p_3$$
such that h_1 and h_2 are diagonal and the Lie brackets with h_1 are given via

1. $[h_1, h_2] = 0$, $[h_1, x] = x$, $[h_1, y] = -y$
2. $[h_1, n_1] = 2n_1, [h_1, n_2] = 0, [h_1, n_3] = n_3$
3. $[h_1, p_1] = -2p_1, [h_1, p_2] = 0, [h_1, p_3] = -p_3$

while the Lie brackets with h_2 are given via

1. $[h_2, x] = -x$, $[h_2, y] = y$
2. $[h_2, n_1] = 0, [h_2, n_2] = 2n_2, [h_2, n_3] = n_3$
3. $[h_2, p_1] = 0, [h_2, p_2] = -2p_2, [h_2, p_3] = -p_3$

Show furthermore that this enumeration has the property that
$$[n_i, n_j] = [p_i, p_j] = 0$$

for all $i, j = 1, 2, 3$, while

$$[n_1, p_1] = h_1, \quad [n_2, p_2] = h_2, \quad [n_3, p_3] = h_1 + h_2, \quad [x, y] = h_1 - h_2$$

Finally, show that

1. $[n_1, p_2] = [n_2, p_1] = 0$
2. $[n_1, p_3] = [n_3, p_2] = x$
3. $[n_2, p_3] = [n_3, p_1] = y$

Conclude that $\mathrm{sp}(4, K)$ is equal to its derived subalgebra. (25 points)

Problem 4: Suppose that K is a field and that $B \in \mathrm{GL}(n, K)$ is an invertible $n \times n$-matrix with entries in K. We have seen in Problem 1 on Sheet 1 that

$$L := \{A \in M(n \times n, K) \mid A^T B = -BA\}$$

is a Lie subalgebra of $\mathrm{gl}(n, K)$. If $C \in M(n \times n, K)$ satisfies $C^T BC = B$, show that C is invertible and that

$$\phi : L \to L, \ A \mapsto CAC^{-1}$$

is a (well-defined) automorphism of L. (25 points)

Due date: Thursday, January 25, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.