Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 1**

Lie Algebras

Problem 1:

1. Suppose that V is a vector space over the field K and that
$$\langle \cdot, \cdot \rangle \colon V \times V \to K$$
is a bilinear form on V. Show that
$$L := \{f \in \mathrm{End}(V) \mid \langle f(v), w \rangle = -\langle v, f(w) \rangle \text{ for all } v, w \in V\}$$
is a Lie subalgebra of $\mathrm{gl}(V) := \mathrm{End}(V)$. (5 points)

2. Suppose that $B \in M(n \times n, K)$ is an $n \times n$-matrix with entries in K. Show that
$$L' := \{A \in M(n \times n, K) \mid A^T B = -BA\}$$
is a Lie subalgebra of $\mathrm{gl}(n, K) := M(n \times n, K)$. (5 points)

3. In the situation of the first part, assume that V is finite-dimensional and that $v_1, \ldots, v_n$ is a basis of V. Let $B := (\langle v_i, v_j \rangle)_{i,j=1,\ldots,n}$ be the fundamental matrix of the bilinear form. Show that the map $L \to L'$ that assigns to each endomorphism its matrix representation with respect to the given basis is a (well-defined) Lie algebra isomorphism between L and L'. (15 points)

Problem 2:

1. Suppose that $B \in M(n \times n, \mathbb{R})$ is an $n \times n$-matrix with entries in the real numbers $\mathbb{R}$. Show that
$$G := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T B A = B\}$$
is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. (5 points)

2. Suppose that $\gamma : \mathbb{R} \to G,\ t \mapsto \gamma(t)$ is a differentiable curve with $\gamma(0) = E_n$, the unit matrix. Show that the derivative $A := \gamma'(0)$ is contained in the Lie algebra L' considered in the second part of Problem 1. (20 points)

(Remarks: A matrix-valued curve is differentiable if and only if each of its matrix entries is differentiable as a function of t. If you write out all matrix products in components, this problem requires only calculus of a single variable. It is slightly more complicated to show that every element of L' can be realized in the form $A := \gamma'(0)$ in this way. The proof of this fact uses a version of the exponential function for matrices. This problem provides an example for the statement that the Lie algebra of a Lie group is its tangent space at the unit element.)

Problem 3: Suppose that V is a vector space over the field K of finite even dimension $\dim(V) = 2n$, and that $\langle \cdot, \cdot \rangle \colon V \times V \to K$ is a bilinear form on V. Assume that $\operatorname{char}(K) \neq 2$, i.e., that $2 := 1+1 \neq 0$ in K. Show that the following conditions are equivalent:

1. There is a basis of V so that the fundamental matrix of the bilinear form is
$$\begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}$$
(four blocks of size $n \times n$).

2. There is a basis of V so that the fundamental matrix of the bilinear form is
$$\begin{pmatrix} X & 0 & 0 & \dots & 0 \\ 0 & X & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & X & 0 \\ 0 & 0 & \dots & 0 & X \end{pmatrix}$$
(n^2 blocks of size 2×2), with $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. There is a basis of V so that the fundamental matrix of the bilinear form is
$$\begin{pmatrix} 2E_n & 0 \\ 0 & -2E_n \end{pmatrix}$$
(a diagonal matrix with n entries 2 followed by n entries -2).

(Remark: Of course, the bases in the three cases are different.) (25 points)

Problem 4: For a field K, the elements

$$x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of $\mathrm{sl}(2, K)$. Show that

$$[h, x] = 2x \qquad [h, y] = -2y \qquad [x, y] = h$$

and explain how the other six Lie brackets of the basis elements follow from these equations and the Lie algebra axioms. (25 points)

Due date: Tuesday, January 16, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.