**Memorial University of Newfoundland** **Winter Semester 2022**
**Yorck Sommerhäuser** **MATH 6329: Sheet 8**

# Hopf Algebras

**Problem 1:** Suppose that $H$ is a quasitriangular Hopf algebra and that $u$ is its Drinfel'd element.

1. Show that $S^2(u) = u$. (5 points)
2. Show that $g := u(S(u))^{-1}$ is group-like. (10 points)
3. Show that $S^4(h) = ghg^{-1}$ for all $h \in H$. (5 points)

**Problem 2:** Suppose that $G$ is a finite group and that $H = K[G]^*$ is the dual group ring considered in Problem 1 on Sheet 3. As there, we denote the dual basis elements of the group elements by $p_g$. Let

$$R = \sum_{g,h \in G} \theta(g,h) p_g \otimes p_h \in H \otimes H$$

be any tensor. Show that $H$ is quasitriangular with universal R-matrix $R$ if and only if $G$ is abelian and $\theta$ is a bicharacter, i.e., a map from $G \times G$ to the multiplicative group $K^\times$ that satisfies

$$\theta(gg', h) = \theta(g,h)\theta(g',h) \qquad \text{and} \qquad \theta(g, hh') = \theta(g,h)\theta(g,h')$$

for all $g, g', h, h' \in G$. (25 points)

**Problem 3:** Suppose that $H$ is a finite-dimensional Hopf algebra. Show that

$$R := \sum_{i=1}^{n} (\varepsilon \otimes h_i) \otimes (h_i^* \otimes 1_H)$$

where $h_1, \ldots, h_n$ is a basis of $H$ with dual basis $h_1^*, \ldots, h_n^*$, is an R-matrix for the Drinfel'd double $D(H)$ of $H$. (25 points)

(Hint: Explain why it is sufficient to show the equation

$$\Delta^{\mathrm{cop}}(\varphi \otimes h) = R\Delta(\varphi \otimes h)R^{-1}$$

in the special cases $h = 1_H$ and $\varphi = \varepsilon$. We have done the first case in class already, so you only need to do the second one. Also, we have already seen in class that $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$, so you only need to show that $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$.)

**Problem 4:** Suppose that $H$ is a finite-dimensional Hopf algebra. If $\lambda \in H^*$ is a left integral and $\Gamma \in H$ is a right integral, show that $\lambda \otimes \Gamma$ is a left integral of the Drinfel'd double of $H$. (30 points)

(Remark: Recall that we have seen in class that this element is also a right integral, so that the Drinfel'd double is unimodular.)

Due date: Tuesday, March 22, 2022. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.