Memorial University of Newfoundland **Winter Semester 2022**
Yorck Sommerhäuser **MATH 6329: Sheet 6**

Hopf Algebras

Problem 1: Over the base field K, consider the algebra H with generators c and s and defining relations

$$c^2 + s^2 = 1 \qquad cs = sc$$

1. Show that there is an algebra homomorphism $\Delta : H \to H \otimes H$ that satisfies
$$\Delta(c) = c \otimes c - s \otimes s \qquad \Delta(s) = c \otimes s + s \otimes c$$
(10 points)

2. Show that this algebra homomorphism Δ is the coproduct of a Hopf algebra structure. For the counit ε and the antipode S of this Hopf algebra structure, find the values $\varepsilon(c)$, $\varepsilon(s)$, $S(c)$, and $S(s)$. (15 points)

3. Show that H is cocommutative. (5 points)

(Hint: In the second part, you need to verify all the Hopf algebra axioms, for example coassociativity and counitality. Make sure that you do not forget any property. Recall that counit and antipode are unique; so if you have found values on the generators that work, these will be the only possibility. The letters c and s are chosen to be reminiscent of cosine and sine.)

Problem 2: Suppose that the base field K contains a primitive fourth root of unity ι.

1. Show that, in the Hopf algebra H considered in Problem 1, the element $b := c + \iota s$ is group-like. (10 points)

2. If $G \cong \mathbb{Z}$ is an infinite cyclic group with generator g, show that $H \cong K[G]$ as Hopf algebras. (10 points)

(Hint: If the field K contains a primitive fourth root of unity ι, then its characteristic is different from 2. Why?)

Problem 3:

1. If $G \cong \mathbb{Z}$ is an infinite cyclic group with generator g, show that the invertible elements of $K[G]$ are exactly the nonzero scalar multiples of powers of g. (15 points)

2. If K contains a primitive fourth root of unity ι, find the invertible elements of the Hopf algebra H considered in the first two problems. (5 points)

Problem 4:

1. Show that for any commutative and cocommutative Hopf algebra H, so especially for the one considered in the first two problems, the map

$$t : H \to H, \; h \mapsto h_{(1)}h_{(2)}$$

is a Hopf algebra homomorphism. (10 points)

2. If the base field K contains a primitive fourth root of unity ι, show that the Hopf algebra H considered in the first two problems is free over the Hopf subalgebra $t(H)$. (20 points)

(Hint: Consider the analogue of the map t for the group algebra $K[G]$, where $G \cong \mathbb{Z}$ is an infinite cyclic group with generator g, and use the isomorphism constructed in Problem 2.2.)

Due date: Tuesday, March 8, 2022. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.