Hopf Algebras

Problem 1: Suppose that $q \in K$ is an element of the base field that is different from 0 and 1. For a natural number $n \in \mathbb{N}$, we define the q-number

$$(n)_q := \frac{q^n - 1}{q - 1} = \sum_{k=0}^{n-1} q^k$$

Furthermore, we define the q-factorial as $(n)_q! := (1)_q (2)_q (3)_q \ldots (n)_q$, with the convention that $(0)_q! := 1$. It may very well happen that q-numbers and q-factorials are zero; this happens if and only if q is a root of unity. Even in this case, we define the q-binomial coefficient, or Gaussian binomial coefficient, as

$$\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}$$

as long as $0 \leq k \leq n$ and the denominator of this expression is nonzero. However, we have in any case by definition that

$$\binom{n}{0}_q = \binom{n}{n}_q := 1$$

even if $(n)_q! = 0$. Furthermore, we define $\binom{0}{0}_q := 1$.

1. For $n \geq 2$ and $k = 1, \ldots, n-1$, show the q-Pascal identity

$$\binom{n}{k}_q = q^{n-k} \binom{n-1}{k-1}_q + \binom{n-1}{k}_q$$

2. Suppose that A is an algebra and that $a, b \in A$ q-commute in the sense that $ba = qab$. Show the q-binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}$$

3. In this situation, suppose that q is a primitive n-th root of unity. Show that

$$(a+b)^n = a^n + b^n$$

(30 points)

Problem 2: Suppose that $q \in K$ is a primitive n-th root of unity. Let T be the algebra generated by g and x subject to the relations

$$g^n = 1 \qquad x^n = 0 \qquad xg = qgx$$

1. Show that there exist unique algebra homomorphisms $\Delta : T \to T \otimes T$ and $\varepsilon : T \to K$ such that

   $$\Delta(g) = g \otimes g \qquad \Delta(x) = 1 \otimes x + x \otimes g \qquad \varepsilon(g) = 1 \qquad \varepsilon(x) = 0$$

   and a unique algebra antihomomorphism $S : T \to T$ satisfying

   $$S(g) = g^{n-1} \qquad S(x) = -xg^{n-1}$$

   (Note that $g^{n-1} = g^{-1}$, the inverse of g.) (10 points)

2. Show that Δ, ε, and S make T into a Hopf algebra, the so-called Taft algebra. (15 points)

Problem 3: Let C be a cyclic group of order n with generator c, and let $K[t]/(t^n)$ be the quotient of the polynomial algebra $K[t]$ by the principal ideal (t^n). We denote the residue class of t by τ. The elements $c^i \otimes \tau^j$ for indices in the range $i, j = 0, \ldots, n-1$ form a basis of $K[C] \otimes K[t]/(t^n)$. We define a product on this space by defining it on basis elements via

$$(c^i \otimes \tau^j)(c^k \otimes \tau^l) = q^{jk} c^{i+k} \otimes \tau^{j+l}$$

and extending bilinearly.

1. Show that this product is associative and that $1 \otimes 1 = c^0 \otimes \tau^0$ is a unit element. (10 points)
2. Show that $K[C] \otimes K[t]/(t^n)$ is isomorphic to T. (10 points)
3. Conclude that the elements $g^i x^j$ for $i, j = 0, \ldots, n-1$ form a basis of T. (5 points)

Problem 4: Suppose that H is a Hopf algebra with antipode S. Suppose furthermore that H^{op} is also a Hopf algebra with antipode S'. Show that $S \circ S' = \mathrm{id}_H = S' \circ S$. (20 points)

Due date: Tuesday, February 1, 2022. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.