Memorial University of Newfoundland **Winter Semester 2022**
Yorck Sommerhäuser **MATH 6329: Sheet 1**

Hopf Algebras

Problem 1: Suppose that V and W are vector spaces over the field K. Show that every tensor $x \in V \otimes W$ can be written as a sum

$$x = \sum_{i=1}^{k} v_i \otimes w_i$$

of decomposable tensors $v_i \otimes w_i$ with the property that both $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are linearly independent. (20 points)

Problem 2: Suppose that V and W are vector spaces over the field K.

1. If $(v_i)_{i \in I}$ is a basis of V, where I is a not necessarily finite index set, show that every tensor $x \in V \otimes W$ can be written uniquely in the form

   $$x = \sum_{i \in I} v_i \otimes w_i$$

   where only finitely many w_i are nonzero. (20 points)

2. If in addition $(w_j)_{j \in J}$ is a basis of W, where J is a not necessarily finite index set, show that every tensor $x \in V \otimes W$ can be written uniquely in the form

   $$x = \sum_{i \in I} \sum_{j \in J} \lambda_{ij} v_i \otimes w_j$$

   where only finitely many coefficients $\lambda_{ij} \in K$ are nonzero. (10 points)

Problem 3: Suppose that V and W are vector spaces over the field K, and that $V^* = \mathrm{Hom}_K(V, K)$ and $W^* = \mathrm{Hom}_K(W, K)$ are the corresponding dual spaces. If $f \in V^*$ and $g \in W^*$, the mapping

$$V \times W \to K, \ (v, w) \mapsto f(v)g(w)$$

is bilinear, and therefore induces a linear map $V \otimes W \to K$, i.e., an element of the dual space $(V \otimes W)^*$, which is, in an ambiguous fashion, also denoted by $f \otimes g$.

1. Show that the map

$$V^* \times W^* \to (V \otimes W)^*, \ (f, g) \mapsto f \otimes g$$

   is bilinear and therefore induces a canonical map

$$c : V^* \otimes W^* \to (V \otimes W)^*$$

   satisfying $c(f \otimes g) = f \otimes g$. (Recall that the notation $f \otimes g$ is ambiguous.) (10 points)

2. Show that c is injective. (15 points)

Problem 4:

1. Show that c is surjective if V or W are finite-dimensional. (10 points)

2. Show that c is not surjective if both V and W are infinite-dimensional. (15 points)

Due date: Tuesday, January 25, 2022. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.