Memorial University of Newfoundland **Winter Semester 2020**
Yorck Sommerhäuser **MATH 6329: Sheet 7**

Hopf Algebras

Problem 1: Suppose that K is a field whose characteristic is different from 2. Let H be the algebra with generators x, y, z subject to the relations

$$x^2 = y^2 = 1 \qquad z^2 = \frac{1}{2}(1 + x + y - xy) \qquad xy = yx \qquad xz = zy \qquad yz = zx$$

1. Show that there are unique algebra homomorphisms

$$\Delta : H \to H \otimes H \qquad \varepsilon : H \to K \qquad S : H \to H^{\text{op}}$$

with the property that

$$\Delta(x) = x \otimes x \qquad \Delta(y) = y \otimes y \qquad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$

as well as $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$ and $S(x) = x$, $S(y) = y$, $S(z) = z$.

2. Show that these structures make H into a Hopf algebra. (25 points)

Problem 2: For the Hopf algebra in Problem 1, define

$$\Lambda := (1 + x)(1 + y)(1 + z)$$

1. Show that Λ is a two-sided integral.
2. Deduce that H is semisimple. (20 points)

Problem 3: Suppose that K is a field whose characteristic is different from 2 and that $\iota \in K$ is a primitive fourth root of unity. Consider the Hopf algebra H defined in Problem 1.

1. Show that there are four algebra homomorphisms $\omega_0, \omega_1, \omega_2, \omega_3$ from H to K, of which the first is the counit $\omega_0 = \varepsilon$ defined in Problem 1 and the remaining three are given on generators by

$$\begin{array}{lll} \omega_1(x) = 1 & \omega_1(y) = 1 & \omega_1(z) = -1 \\ \omega_2(x) = -1 & \omega_2(y) = -1 & \omega_2(z) = \iota \\ \omega_3(x) = -1 & \omega_3(y) = -1 & \omega_3(z) = -\iota \end{array}$$

2. Show that there are no other algebra homomorphisms from H to K.
3. Show that the group of group-like elements $G(H^*)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (15 points)

Problem 4: We continue to use the assumptions and notations from Problem 3.

1. Show that there is a representation $\rho : H \to M(2\times 2, K)$ with the property that

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. Show that ρ is irreducible.
3. Show that every two-dimensional irreducible representation is isomorphic to ρ.
4. Show that the elements $x^i y^j z^k$ for $i, j, k \in \{0, 1\}$ form a basis of H. Use this to compute the dimension of H. (40 points)

Due date: There is no due date. The completion of these problems is voluntary. The solutions will not be collected and not be marked, unless explicitly requested otherwise.