Memorial University of Newfoundland **Winter Semester 2020**
Yorck Sommerhäuser **MATH 6329: Sheet 4**

Hopf Algebras

Problem 1: Suppose that H is a Hopf algebra with bijective antipode, and that V is left module over H.

1. Show that V^* is again a left H-module with respect to the action
$$(h.\varphi)(v) := \varphi(S^{-1}(h).(v))$$
(10 points)

2. Show that the evaluation map
$$\mathrm{ev}' : V \otimes V^* \to K,\ v \otimes \varphi \mapsto \varphi(v)$$
is H-linear. (5 points)

3. If V is finite-dimensional with basis $v_1, \ldots, v_n$ and dual basis $v_1^*, \ldots, v_n^*$, show that dual basis map
$$\mathrm{db}' : K \to V^* \otimes V,\ \lambda \mapsto \lambda \sum_{i=1}^{n} v_i^* \otimes v_i$$
is H-linear. (10 points)

Problem 2: Suppose that H is a Hopf algebra with bijective antipode, and that V is a finite-dimensional left comodule over H with basis $v_1, \ldots, v_n$ and dual basis $v_1^*, \ldots, v_n^*$.

1. Show that the dual space is again a left comodule over H when endowed with the coaction
$$\delta_{V^*} : V^* \to H \otimes V^*,\ \varphi \mapsto \sum_{i=1}^{n} S^{-1}(v_i^{(1)})\varphi(v_i^{(2)}) \otimes v_i^*$$
(10 points)

2. Show that this coaction satisfies
$$\varphi^{(1)}\varphi^{(2)}(v) = S^{-1}(v^{(1)})\varphi(v^{(2)})$$
for all $\varphi \in V^*$ and all $v \in V$. (5 points)

3. Show that the evaluation map

$$\mathrm{ev} : V^* \otimes V \to K, \ \varphi \otimes v \mapsto \varphi(v)$$

is colinear. (10 points)

Problem 3: Let L be a Lie algebra. An associative algebra U (with unit) together with a linear map $\iota : L \to U$ is called a universal enveloping algebra if

$$\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$$

for all $x, y \in L$ and the following universal property holds: If A is another associative algebra (with unit) together with a linear map $i : L \to A$ satisfying

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

for all $x, y \in L$, then there is a unique (unit-preserving) algebra homomorphism $f : U \to A$ satisfying $f \circ \iota = i$.

1. Deduce from the properties stated above that the subspace $\iota(L)$ generates U as an algebra.
2. Show that there is a unique Hopf algebra structure on U with the following properties:
   (a) $\Delta(\iota(x)) = \iota(x) \otimes 1 + 1 \otimes \iota(x)$ for all $x \in L$.
   (b) $\varepsilon(\iota(x)) = 0$ for all $x \in L$.
   (c) $S(\iota(x)) = -\iota(x)$ for all $x \in L$.

(Remark: It is a consequence of the (nontrivial) Poincaré-Birkhoff-Witt theorem that ι is injective. This fact is not needed for this problem.) (25 points)

Problem 4: Decide whether the Taft algebras T defined on Sheet 2 are semisimple. If semisimplicity depends on properties of the parameter q, find and state these properties. Prove all your assertions in complete detail. (25 points)

Due date: There is no due date. The completion of these problems is voluntary. The solutions will not be collected and not be marked, unless explicitly requested otherwise.