Memorial University of Newfoundland **Winter Semester 2020**
Yorck Sommerhäuser **MATH 6329: Sheet 3**

Hopf Algebras

Problem 1: Suppose that G is a finite group. In the dual $K[G]^*$ of the group algebra, we have the dual basis of the basis consisting of the elements of G, which we denote by p_g, so that $p_g(h) = \delta_{g,h}$ for all elements $g, h \in G$. Show that the Hopf algebra structure of $K[G]^*$ is given on this basis by the formulas

1. $\Delta(p_g) = \sum_{\substack{h,h' \in G \\ hh'=g}} p_h \otimes p_{h'}$
2. $\varepsilon(p_g) = \delta_{g,1}$
3. $p_g p_h = \delta_{g,h} p_g$
4. $S(p_g) = p_{g^{-1}}$ (20 points)

(Remark: Since a linear functional is uniquely determined by its values on a basis, this Hopf algebra can also be constructed as the algebra of functions on the group. The multiplication is then the pointwise multiplication, and the elements p_g become the characteristic functions on the singletons.)

Problem 2: Suppose that C is a cyclic finite group of order n with generator c, and suppose that $\zeta \in K$ is a primitive n-th root of unity.

1. Show that there is a unique algebra homomorphism $\chi : K[C] \to K$ with the property that $\chi(c) = \zeta$. (5 points)
2. Show that $\chi \in K[C]^*$ is group-like. (10 points)
3. Show that there is a unique Hopf algebra isomorphism $f : K[C] \to K[C]^*$ with the property that $f(c) = \chi$. (15 points)

Problem 3: Suppose that G is a finite group.

1. Show that if G is abelian, then $K[G]$ and $K[G]^*$ are isomorphic as Hopf algebras. (15 points)
2. Show that if G is not abelian, then $K[G]$ and $K[G]^*$ are not isomorphic as Hopf algebras. (15 points)

Problem 4: Suppose that H is a Hopf algebra. An algebra A that is simultaneously an H-module is called a module algebra if

$$h.(ab) = (h_{(1)}.a)(h_{(2)}.b) \qquad \text{and} \qquad h.1_A = \varepsilon_H(h)1_A$$

where the dot indicates the module action.

1. Show that H is a module algebra over itself with respect to the left adjoint action
$$h.h' := h_{(1)}h'S(h_{(2)})$$
2. Show that this module structure satisfies the so-called Yetter-Drinfel'd condition
$$\Delta(h.h') := h_{(1)}h'_{(1)}S(h_{(3)}) \otimes h_{(2)}.h'_{(2)}$$

(20 points)

Due date: There is no due date. The completion of these problems is voluntary. The solutions will not be collected and not be marked, unless explicitly requested otherwise.