Memorial University of Newfoundland **Winter Semester 2016**
Yorck Sommerhäuser **MATH 6329: Sheet 9**

Hopf Algebras

Problem 1: Suppose that K is a field whose characteristic is different from 2. Let H be the algebra with generators x, y, z subject to the relations

$$x^2 = y^2 = 1 \qquad z^2 = \frac{1}{2}(1 + x + y - xy) \qquad xy = yx \qquad xz = zy \qquad yz = zx$$

1. Show that there are unique algebra homomorphisms

$$\Delta : H \to H \otimes H \qquad \varepsilon : H \to K \qquad S : H \to H^{\text{op}}$$

   with the property that

$$\Delta(x) = x \otimes x \qquad \Delta(y) = y \otimes y \qquad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$

   as well as $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$ and $S(x) = x$, $S(y) = y$, $S(z) = z$.

2. Show that these structures make H into a Hopf algebra. (5 points)

Problem 2: For the Hopf algebra in Problem 1, define

$$\Lambda := (1 + x)(1 + y)(1 + z)$$

1. Show that Λ is a two-sided integral.
2. Deduce that H is semisimple. (4 points)

Problem 3: Suppose that K is a field whose characteristic is different from 2 and that $\iota \in K$ is a primitive fourth root of unity. Consider the Hopf algebra H defined in Problem 1.

1. Show that there are four algebra homomorphisms $\omega_0, \omega_1, \omega_2, \omega_3$ from H to K, of which the first is the counit $\omega_0 = \varepsilon$ defined in Problem 1 and the remaining three are given on generators by

$$\begin{array}{lll} \omega_1(x) = 1 & \omega_1(y) = 1 & \omega_1(z) = -1 \\ \omega_2(x) = -1 & \omega_2(y) = -1 & \omega_2(z) = \iota \\ \omega_3(x) = -1 & \omega_3(y) = -1 & \omega_3(z) = -\iota \end{array}$$

2. Show that there are no other algebra homomorphisms from H to K.
3. Show that the group of group-like elements $G(H^*)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (3 points)

Problem 4: We continue to use the assumptions and notations from Problem 3.

1. Show that there is a representation $\rho : H \to M(2\times 2, K)$ with the property that

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. Show that ρ is irreducible.
3. Show that every two-dimensional irreducible representation is isomorphic to ρ.
4. Show that the elements $x^i y^j z^k$ for $i, j, k \in \{0, 1\}$ form a basis of H. Use this to compute the dimension of H. (8 points)

Due date: Thursday, March 31, 2016. Please write your solution on letter-sized paper, and write your name on your solution. Give all your computations in complete detail, and explain these computations in English, using complete sentences. It is not necessary to copy down the problems again, and it is also not necessary to submit this sheet with your solution.