Memorial University of Newfoundland **Winter Semester 2016**
Yorck Sommerhäuser **MATH 6329: Sheet 8**

Hopf Algebras

Problem 1: Suppose that H is a Hopf algebra over K and that $L \supset K$ is a field extension of K. Then L is in particular an algebra over K, so that the tensor product algebra $H_L := H \otimes_K L$ is an algebra over K.

1. Show that H_L is an algebra over L if we define multiplication by scalars from L via the equation

$$\lambda(a \otimes_K \mu) := a \otimes_K \lambda\mu$$

for $a \in A$ and $\lambda, \mu \in L$. (1 point)

2. Show that H_L is a coalgebra over L with respect to the coproduct Δ_L defined as the composition

$$H \otimes_K L \xrightarrow{\Delta \otimes_K \mathrm{id}_L} H \otimes_K H \otimes_K L \xrightarrow{\cong} (H \otimes_K L) \otimes_L (H \otimes_K L)$$

and the counit $\varepsilon_L(h \otimes_K \lambda) = \varepsilon(h)\lambda$. Explain the second arrow in the composition above. (1 point)

3. Show that is a Hopf algebra over L with the antipode $S_L := S \otimes_K \mathrm{id}_L$. (1 point)

Problem 2: In the situation of Problem 4 on Sheet 7, show that H is not free over $t(H)$ if the characteristic of K is different from 2 and K does not contain a primitive fourth root of unity.
(Hint: Use that, over a commutative ring, every basis of a module has the same number of elements. Consider the field extension $L \supset K$ that arises by adjoining a fourth root of unity ι to K. The extension $L \supset K$ is a quadratic Galois extension with a Galois group that consists of two elements. If we denote the element of the Galois group that is different from the identity by σ, consider the ring homomorphism

$$H_L \to H_L,\ h \otimes \mu \mapsto h \otimes \sigma(\mu)$$

which we also denote by σ. A basis of H over $t(H)$ leads to a basis of H_L over $t(H_L)$. Another basis of H_L over $t(H_L)$ was constructed in Problem 4.2 on Sheet 7. If b is the element considered in Problem 2.1 on Sheet 7, show that the determinant a of the base change matrix satisfies the equation $\sigma(a) = ab^2$, because every entry in the second column of the matrix satisfies this equation, while those in the first column are invariant under σ. Using the fact that the determinant of an invertible matrix is invertible, together with the description of the invertible elements in Problem 3 on Sheet 7, derive a contradiction. You can assume that $K = \mathbb{R}$, $L = \mathbb{C}$, $\iota = i$, and that σ is complex conjugation if you like. This example is due to U. Oberst and H.-J. Schneider.) (7 points)

Problem 3: If A is an algebra, an element $c = \sum_{i=1}^{m} x_i \otimes y_i$ is called a Casimir element if

$$\sum_{i=1}^{m} a x_i \otimes y_i = \sum_{i=1}^{m} x_i \otimes y_i a$$

for all $a \in A$. The Casimir element is called symmetric if

$$\sum_{i=1}^{m} x_i \otimes y_i = \sum_{i=1}^{m} y_i \otimes x_i$$

1. Show that the set of Casimir elements and the set of symmetric Casimir elements are subspaces of $A \otimes A$. (1 point)
2. If $A = M(n \times n, K)$ is the algebra of $n \times n$-matrices, show that
$$c = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}$$
is a symmetric Casimir element, where E_{ij} is the (i,j)-th matrix unit. (1 point)
3. If $A = M(n \times n, K)$, show that every symmetric Casimir element is a scalar multiple of c. (2 points)
4. Find all Casimir elements in the algebra $A = M(n \times n, K)$, and determine the dimension of the space of Casimir elements. (2 points)

Problem 4: Show that a separable algebra is finite-dimensional. Conclude that a semisimple Hopf algebra is finite-dimensional. (Hint: Be careful not to get into circular arguments.) (4 points)

Thursday, March 17, 2016. Please write your solution on letter-sized paper, and write your name on your solution. Give all your computations in complete detail, and explain these computations in English, using complete sentences. It is not necessary to copy down the problems again, and it is also not necessary to submit this sheet with your solution.