Hopf Algebras

Problem 1: Suppose that A is a bialgebra, and that V and W are left comodules over A.

1. Show that $V \otimes W$ is again a comodule when endowed with the coaction
$$\delta_{V\otimes W}(v \otimes w) := v^{(1)}w^{(1)} \otimes v^{(2)} \otimes w^{(2)}$$
2. If U is another left comodule over A, show that the canonical isomorphism of vector spaces
$$(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$$
is colinear.
3. Show that the base field K becomes a comodule over A when endowed with the coaction
$$\delta_K : K \to A \otimes K,\ \lambda \mapsto 1 \otimes \lambda$$
4. Show that the canonical isomorphisms $K \otimes V \to V,\ \lambda \otimes v \mapsto \lambda v$ and $V \otimes K \to V,\ v \otimes \lambda \mapsto \lambda v$ are colinear. (4 points)

Problem 2: Suppose that H is a Hopf algebra, and that V is a finite-dimensional left comodule over H with basis $v_1, \ldots, v_n$ and dual basis $v_1^*, \ldots, v_n^*$.

1. Show that the dual space is again a left comodule over H when endowed with the coaction
$$\delta_{V^*} : V^* \to H \otimes V^*,\ \varphi \mapsto \sum_{i=1}^{n} S(v_i^{(1)})\varphi(v_i^{(2)}) \otimes v_i^*$$
2. Show that this coaction satisfies
$$\varphi^{(1)}\varphi^{(2)}(v) = S(v^{(1)})\varphi(v^{(2)})$$
for all $\varphi \in V^*$ and all $v \in V$.
3. Show that the evaluation map
$$\mathrm{ev} : V \otimes V^* \to K,\ v \otimes \varphi \mapsto \varphi(v)$$
is colinear. (3 points)

Problem 3: Let L be a Lie algebra. An associative algebra U (with unit) together with a linear map $\iota : L \to U$ is called a universal enveloping algebra if

$$\iota([x,y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$$

for all $x, y \in L$ and the following universal property holds: If A is another associative algebra (with unit) together with a linear map $i : L \to A$ satisfying

$$i([x,y]) = i(x)i(y) - i(y)i(x)$$

for all $x, y \in L$, then there is a unique (unit-preserving) algebra homomorphism $f : U \to A$ satisfying $f \circ \iota = i$.

1. Deduce from the properties stated above that the subspace $\iota(L)$ generates U as an algebra.
2. Show that there is a unique Hopf algebra structure on U with the following properties:
   (a) $\Delta(\iota(x)) = \iota(x) \otimes 1 + 1 \otimes \iota(x)$ for all $x \in L$.
   (b) $\varepsilon(\iota(x)) = 0$ for all $x \in L$.
   (c) $S(\iota(x)) = -\iota(x)$ for all $x \in L$.

(Remark: It is a consequence of the (nontrivial) Poincaré-Birkhoff-Witt theorem that ι is injective. This fact is not needed for this problem.) (6 points)

Problem 4: Decide whether the Taft algebras T defined on Exercise Sheet 2 are semisimple. If semisimplicity depends on properties of the parameter q, find and state these properties. Prove all your assertions in complete detail. (7 points)

Due date: Thursday, February 4, 2016. Please write your solution on letter-sized paper, and write your name on your solution. Give all your computations in complete detail, and explain these computations in English, using complete sentences. It is not necessary to copy down the problems again, and it is also not necessary to submit this sheet with your solution.