Hopf Algebras

Problem 1: Suppose that H is a quasitriangular Hopf algebra and that u is its Drinfel'd element.

1. Show that $S^2(u) = u$.   (1 point)

2. Show that $g := u(S(u))^{-1}$ is group-like.   (2 points)

3. Show that $S^4(h) = ghg^{-1}$ for all $h \in H$.   (1 point)

Problem 2: Suppose that H is a quasitriangular Hopf algebra with universal R-matrix R, that u is its Drinfel'd element and that $g := u(S(u))^{-1}$. Define $h := (\mathrm{id} \otimes \alpha^R)(R)$, where α^R is the right modular function.

1. Show that h is group-like.

2. Show that $g = ha^R$, where a^R is the right modular element.   (6 points)

Problem 3: Suppose that G is a finite group and that $H = K[G]^*$ is the dual group ring considered in Problem 1 on Sheet 3. As there, we denote the dual basis elements of the group elements by p_g. Let

$$R = \sum_{g,h \in G} \theta(g,h) p_g \otimes p_h \in H \otimes H$$

be any tensor. Show that H is quasitriangular with universal R-matrix R if and only if G is abelian and θ is a bicharacter, i.e., a map from $G \times G$ to the multiplicative group $K^\times$ that satisfies

$$\theta(gg', h) = \theta(g,h)\theta(g', h) \qquad \text{and} \qquad \theta(g, hh') = \theta(g,h)\theta(g, h')$$

for all $g, g', h, h' \in G$.   (4 points)

Problem 4: Suppose that H is a finite-dimensional quasitriangular Hopf algebra with R-matrix R.

1. If D is a Drinfel'd double of H, show that there exists a Hopf algebra homomorphism $\pi : D \to H$ that satisfies

$$\pi(\kappa(h)) = h \qquad \text{and} \qquad \pi(\nu(\varphi)) = (\varphi \otimes \mathrm{id}_H)(R)$$

for all $h \in H$ and all $\varphi \in H^*$. \hfill (2 points)

2. If H is in addition semisimple and the base field is algebraically closed of characteristic zero, show that the dimension of a simple H-module V divides the dimension of H. (Hint: Use Theorem 6.4.) \hfill (4 points)

Due date: Wednesday, April 6, 2016. Please write your solution on letter-sized paper, and write your name on your solution. Give all your computations in complete detail, and explain these computations in English, using complete sentences. It is not necessary to copy down the problems again, and it is also not necessary to submit this sheet with your solution.