Memorial University of Newfoundland **Fall Semester 2024**
Yorck Sommerhäuser **MATH 6329: Sheet 9**

Hopf Algebras

Problem 1: Suppose that K is a field whose characteristic is different from 2. Let H be the algebra with generators x, y, z subject to the defining relations

$$x^2 = y^2 = 1 \qquad z^2 = \frac{1}{2}(1 + x + y - xy) \qquad xy = yx \qquad xz = zy \qquad yz = zx$$

1. Show that there are unique algebra homomorphisms

$$\Delta : H \to H \otimes H \qquad \varepsilon : H \to K \qquad S : H \to H^{\mathrm{op}}$$

   with the property that

$$\Delta(x) = x \otimes x \qquad \Delta(y) = y \otimes y \qquad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$

   as well as $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$ and $S(x) = x$, $S(y) = y$, $S(z) = z$.

2. Show that these structures make H into a Hopf algebra. (25 points)

Problem 2: For the Hopf algebra in Problem 1, define

$$\Lambda := (1 + x)(1 + y)(1 + z)$$

1. Show that Λ is a two-sided integral.
2. Deduce that H is semisimple. (20 points)

Problem 3: Suppose that K is a field whose characteristic is different from 2 and that $\iota \in K$ is a primitive fourth root of unity. Consider the Hopf algebra H defined in Problem 1.

1. Show that there are four algebra homomorphisms $\omega_0, \omega_1, \omega_2, \omega_3$ from H to K, of which the first is the counit $\omega_0 = \varepsilon$ defined in Problem 1 and the remaining three are given on generators by

$$\begin{array}{lll} \omega_1(x) = 1 & \omega_1(y) = 1 & \omega_1(z) = -1 \\ \omega_2(x) = -1 & \omega_2(y) = -1 & \omega_2(z) = \iota \\ \omega_3(x) = -1 & \omega_3(y) = -1 & \omega_3(z) = -\iota \end{array}$$

2. Show that there are no other algebra homomorphisms from H to K.
3. Show that the group of group-like elements $G(H^*)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (15 points)

Problem 4: We continue to use the assumptions and notations from Problem 3.

1. Show that there is a representation $\rho : H \to M(2\times 2, K)$ with the property that

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. Show that ρ is irreducible.
3. Show that every two-dimensional irreducible representation is isomorphic to ρ.
4. Show that the elements $x^i y^j z^k$ for $i, j, k \in \{0, 1\}$ form a basis of H. Use this to compute the dimension of H. (40 points)

Due date: Thursday, November 14, 2024. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.