**Memorial University of Newfoundland** **Fall Semester 2024**
**Yorck Sommerhäuser** **MATH 6329: Sheet 3**

# Hopf Algebras

**Problem 1:** Suppose that $G$ is a finite group. In the dual $K[G]^*$ of the group algebra, we have the dual basis of the basis consisting of the elements of $G$, which we denote by $p_g$, so that $p_g(h) = \delta_{g,h}$ for all elements $g, h \in G$. Show that the Hopf algebra structure of $K[G]^*$ is given on this basis by the formulas

1. $\Delta(p_g) = \sum_{\substack{h,h' \in G \\ hh'=g}} p_h \otimes p_{h'}$
2. $\varepsilon(p_g) = \delta_{g,1}$
3. $p_g p_h = \delta_{g,h} p_g$
4. $S(p_g) = p_{g^{-1}}$ (20 points)

(Remark: Since a linear functional is uniquely determined by its values on a basis, this Hopf algebra can also be constructed as the algebra of functions on the group. The multiplication is then the pointwise multiplication, and the elements $p_g$ become the characteristic functions on the singletons.)

**Problem 2:** Suppose that $C$ is a cyclic finite group of order $n$ with generator $c$, and suppose that $\zeta \in K$ is a primitive $n$-th root of unity.

1. Show that there is a unique algebra homomorphism $\chi : K[C] \to K$ with the property that $\chi(c) = \zeta$. (5 points)
2. Show that $\chi \in K[C]^*$ is group-like. (10 points)
3. Show that there is a unique Hopf algebra isomorphism $f : K[C] \to K[C]^*$ with the property that $f(c) = \chi$. (15 points)

**Problem 3:** Suppose that $G$ is a finite group and that $K$ is an algebraically closed field of characteristic zero.

1. Show that if $G$ is abelian, then $K[G]$ and $K[G]^*$ are isomorphic as Hopf algebras. (15 points)
2. Show that if $G$ is not abelian, then $K[G]$ and $K[G]^*$ are not isomorphic as Hopf algebras. (15 points)

**Problem 4:** Suppose that $H$ is a Hopf algebra. An algebra $A$ that is simultaneously an $H$-module is called a module algebra if

$$h.(ab) = (h_{(1)}.a)(h_{(2)}.b) \qquad \text{and} \qquad h.1_A = \varepsilon_H(h)1_A$$

where the dot indicates the module action.

1. Show that $H$ is a module algebra over itself with respect to the left adjoint action
$$h.h' := h_{(1)}h'S(h_{(2)})$$
2. Show that this module structure satisfies the so-called Yetter-Drinfel'd condition
$$\Delta(h.h') := h_{(1)}h'_{(1)}S(h_{(3)}) \otimes h_{(2)}.h'_{(2)}$$

(20 points)

Due date: Tuesday, October 1, 2024. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.